

Online Appendix for “Mislearning from Censored Data: The Gambler’s Fallacy in Optimal-Stopping Problems”

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OA 1 Proofs Omitted from the Appendix

OA 1.1 Completing the Proof of of Lemma A.15

Proof. From the hypothesis on g ’s magnitude being bounded, there is some $B < \infty$ so that $0 < g(\mu_1, \mu_2) < B$ for all $\mu_1, \mu_2 \in \mathbb{R}$.

I check conditions A1 through A5 in [Bunke and Milhaud \(1998\)](#). The lemma follows from their Theorem 2 when these conditions are satisfied.

The parameter space is $\Theta = \mathbb{R}^2$. The data-generating density of observation (x, y) is:

$$f^\bullet(x, y) = \begin{cases} \phi(x; \mu_1^\bullet, \sigma^2) \cdot \phi(y; \mu_2^\bullet, \sigma^2) & \text{if } x < c_\dagger \\ \phi(x; \mu_1^\bullet, \sigma^2) \cdot \phi(y; 0, 1) & \text{if } x \geq c_\dagger \end{cases}$$

where $\phi(\cdot; \mu, \sigma^2)$ is the Gaussian density with mean μ and variance σ^2 . Under the subjective model $\Psi(\hat{\mu}_1, \hat{\mu}_2; \gamma)$, the same observation has density:

$$f_{\hat{\mu}_1, \hat{\mu}_2}(x, y) = \begin{cases} \phi(x; \hat{\mu}_1, \sigma^2) \cdot \phi(y; \hat{\mu}_2 - \gamma \cdot (x - \hat{\mu}_1), \sigma^2) & \text{if } x < c_\dagger \\ \phi(x; \hat{\mu}_1, \sigma^2) \cdot \phi(y; 0, 1) & \text{if } x \geq c_\dagger. \end{cases}$$

A1. Parameter space is a closed, convex set in \mathbb{R}^2 with nonempty interior. The density $f_{\hat{\mu}_1, \hat{\mu}_2}(x, y)$ is bounded over $(\hat{\mu}_1, \hat{\mu}_2, x, y)$ and its carrier, $\{(x, y) : f_{\hat{\mu}_1, \hat{\mu}_2}(x, y) > 0\}$ is the same for all $\hat{\mu}_1, \hat{\mu}_2$.

Evidently \mathbb{R}^2 is closed in itself. The density $f_{\hat{\mu}_1, \hat{\mu}_2}(x, y)$ is bounded by the product of the modes of Gaussian densities with variance σ^2 and variance 1. The density $f_{\hat{\mu}_1, \hat{\mu}_2}(x, y)$ is strictly positive on \mathbb{R}^2 for any parameter values $\hat{\mu}_1, \hat{\mu}_2$.

A2. For all $\hat{\mu}_1, \hat{\mu}_2$, there is a sphere $S[(\hat{\mu}_1, \hat{\mu}_2), \eta]$ of center $(\hat{\mu}_1, \hat{\mu}_2)$ and radius $\eta > 0$ such that:

$$\mathbb{E}_{f^\bullet} \left[\sup_{(\mu'_1, \mu'_2) \in S[(\hat{\mu}_1, \hat{\mu}_2), \eta]} \left| \ln \frac{f^\bullet(X, Y)}{f_{\mu'_1, \mu'_2}(X, Y)} \right| \right] < \infty.$$

Pick say $\eta = 1$. Consider the rectangle $R[(\hat{\mu}_1, \hat{\mu}_2), \eta]$ consisting of points (μ'_1, μ'_2) such that $|\mu'_1 - \hat{\mu}_1| < \eta$ and $|\mu'_2 - \hat{\mu}_2| < \eta$. Since the the Gaussian distribution is single-peaked, for any $(x, y) \in \mathbb{R}^2$ the absolute value of the log likelihood ratio $\left| \ln \frac{f^\bullet(X, Y)}{f_{\mu'_1, \mu'_2}(X, Y)} \right|$ on all of $R[(\hat{\mu}_1, \hat{\mu}_2), \eta]$ must be bounded by its value at the 4 corners. That is to say,

$$\begin{aligned} & \sup_{(\mu'_1, \mu'_2) \in S[(\hat{\mu}_1, \hat{\mu}_2), \eta]} \left| \ln \frac{f^\bullet(X, Y)}{f_{\mu'_1, \mu'_2}(X, Y)} \right| \\ & \leq \sup_{(\mu'_1, \mu'_2) \in R[(\hat{\mu}_1, \hat{\mu}_2), \eta]} \left| \ln \frac{f^\bullet(X, Y)}{f_{\mu'_1, \mu'_2}(X, Y)} \right| \\ & \leq \left| \ln \frac{f^\bullet(X, Y)}{f_{\hat{\mu}_1 - \eta, \hat{\mu}_2 - \eta}(X, Y)} \right| + \left| \ln \frac{f^\bullet(X, Y)}{f_{\hat{\mu}_1 - \eta, \hat{\mu}_2 + \eta}(X, Y)} \right| + \left| \ln \frac{f^\bullet(X, Y)}{f_{\hat{\mu}_1 + \eta, \hat{\mu}_2 - \eta}(X, Y)} \right| + \left| \ln \frac{f^\bullet(X, Y)}{f_{\hat{\mu}_1 + \eta, \hat{\mu}_2 + \eta}(X, Y)} \right|. \end{aligned}$$

It is easy to see that for any fixed parameter $\mathbb{E}_{f^\bullet} \left[\left| \ln \frac{f^\bullet(X, Y)}{f_{\mu'_1, \mu'_2}(X, Y)} \right| \right]$ is finite, so the sum of these 4 terms gives a finite upper bound.

A3. For all fixed $(x_0, y_0) \in \mathbb{R}^2$, the map from parameters to density $(\mu_1, \mu_2) \mapsto f_{\mu_1, \mu_2}(x_0, y_0)$ has continuous derivatives with respect to parameters $(\mu_1, \mu_2) \mapsto \frac{\partial f}{\partial \mu_1}(x_0, y_0; \mu_1, \mu_2)$, $(\mu_1, \mu_2) \mapsto \frac{\partial f}{\partial \mu_2}(x, y; \mu_1, \mu_2)$. There exist positive constants κ_0 and b_0 with

$$\int \int \left\| [f_{\mu_1, \mu_2}(x, y)]^{-1} \cdot \begin{pmatrix} \frac{\partial f}{\partial \mu_1}(x, y; \mu_1, \mu_2) \\ \frac{\partial f}{\partial \mu_2}(x, y; \mu_1, \mu_2) \end{pmatrix} \right\|^{12} \cdot f_{\mu_1, \mu_2}(x, y) \cdot dy dx < \kappa_0 (1 + \|(\mu_1, \mu_2)\|^{b_0})$$

satisfied for every $(\mu_1, \mu_2) \in \mathbb{R}^2$, where $\|\cdot\|$ is a norm on \mathbb{R}^2 .

Let's choose the max norm, $\|v\| = \max(|v_1|, |v_2|)$. For uncensored data (x_0, y_0) with $x_0 < c_\dagger$, we can compute

$$\frac{\partial f}{\partial \mu_1}(x_0, y_0; \mu_1, \mu_2) = f_{\mu_1, \mu_2}(x_0, y_0) \cdot \left[\frac{(1 + \gamma^2)}{\sigma^2} \cdot (x - \mu_1) + \frac{\gamma}{\sigma^2} \cdot (y - \mu_2) \right]$$

and

$$\frac{\partial f}{\partial \mu_2}(x_0, y_0; \mu_1, \mu_2) = f_{\mu_1, \mu_2}(x_0, y_0) \cdot \left[\frac{\gamma}{\sigma^2} \cdot (x - \mu_1) - \frac{1}{\sigma^2} \cdot (y - \mu_2) \right].$$

While for censored data (x_0, y_0) where $x_0 > c_\dagger$, the likelihood of the data is unchanged by parameter μ_2 since it neither changes the distribution of the early draw quality nor the distribution of the white noise term, meaning $\frac{\partial f}{\partial \mu_2}(x_0, y_0; \mu_1, \mu_2) = 0$. Also, for the censored case

$$\frac{\partial f}{\partial \mu_1}(x_0, y_0; \mu_1, \mu_2) = f_{\mu_1, \mu_2}(x_0, y_0) \cdot \frac{1}{\sigma^2} (x - \mu_1).$$

This means the integral to be bounded is:

$$\int_{x=-\infty}^{x=c_{\dagger}} \left[\int_{-\infty}^{\infty} \left\| \begin{pmatrix} \frac{(1+\gamma^2)}{\sigma^2} \cdot (x - \mu_1) + \frac{\gamma}{\sigma^2} \cdot (y - \mu_2) \\ \frac{\gamma}{\sigma^2} \cdot (x - \mu_1) - \frac{1}{\sigma^2} \cdot (y - \mu_2) \end{pmatrix} \right\|^{12} \cdot f_{\mu_1, \mu_2}(x, y) \cdot dy \right] dx \\ + \int_{x=c_{\dagger}}^{x=\infty} \left[\int_{-\infty}^{\infty} \left(\frac{1}{\sigma^2} (x - \mu_1) \right)^{12} \cdot f_{\mu_1, \mu_2}(x, y) \cdot dy \right] dx.$$

Since the inner integrals are non-negative, this expression is smaller than the version where the domains of the outer integrals are expanded and the densities $f_{\mu_1, \mu_2}(x, y)$ are simply replaced with the joint density on \mathbb{R}^2 of the subjective model for $\Psi(\mu_1, \mu_2; \gamma)$, which I denote as $\tilde{f}_{\mu_1, \mu_2}(x, y)$.

$$\int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \left\| \begin{pmatrix} \frac{(1+\gamma^2)}{\sigma^2} \cdot (x - \mu_1) + \frac{\gamma}{\sigma^2} \cdot (y - \mu_2) \\ \frac{\gamma}{\sigma^2} \cdot (x - \mu_1) - \frac{1}{\sigma^2} \cdot (y - \mu_2) \end{pmatrix} \right\|^{12} \cdot \tilde{f}_{\mu_1, \mu_2}(x, y) \cdot dy \right] dx \\ + \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \left(\frac{1}{\sigma^2} (x - \mu_1) \right)^{12} \cdot \tilde{f}_{\mu_1, \mu_2}(x, y) \cdot dy \right] dx.$$

The second summand is a 12th moment of the joint normal random variable with distribution $\Psi(\mu_1, \mu_2; \gamma)$, so for all μ_1, μ_2 it is given by some 12th order polynomial $P_2(\mu_1, \mu_2)$. Similarly the first summand is also given by a 12th order polynomial $P_1(\mu_1, \mu_2)$. Therefore by choosing $b_0 = 12$ and choosing κ_0 appropriately according to the coefficients in P_1 and P_2 , we achieved the desired bound.

A4. For some positive constants b_1 and κ_1 , the affinity function

$$A(\mu_1, \mu_2) := \int \int [f_{\mu_1, \mu_2}(x, y) \cdot f^{\bullet}(x, y)]^{1/2} dy dx$$

satisfies $A(\mu_1, \mu_2) < \kappa_1 \cdot \|(\mu_1, \mu_2)\|^{-b_1}$ for all μ_1, μ_2 .

We have $A(\mu_1, \mu_2) \leq [\int \int [f_{\mu_1, \mu_2}(x, y) \cdot f^{\bullet}(x, y)] dy dx]^{1/2}$, so it's sufficient to find some κ_1

and b_1 that works to bound $\int \int [f_{\mu_1, \mu_2}(x, y) \cdot f^\bullet(x, y)] dy dx$. We have:

$$\begin{aligned}
& \int \int [f_{\mu_1, \mu_2}(x, y) \cdot f^\bullet(x, y)] dy dx \\
&= \int_{x=-\infty}^{c_\dagger} \int_{-\infty}^{\infty} \phi(x; \mu_1, \sigma^2) \cdot \phi(x; \mu_1^\bullet, \sigma^2) \cdot \phi(y; \mu_2 - \gamma(x - \mu_1), \sigma^2) \cdot \phi(y; \mu_2^\bullet, \sigma^2) dy dx \\
&+ \int_{x=c_\dagger}^{\infty} \int_{-\infty}^{\infty} \phi(x; \mu_1, \sigma^2) \cdot \phi(x; \mu_1^\bullet, \sigma^2) \cdot \phi(y; 0, 1) \cdot \phi(y; 0, 1) dy dx \\
&\leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(x; \mu_1, \sigma^2) \cdot \phi(x; \mu_1^\bullet, \sigma^2) \cdot \phi(y; \mu_2 - \gamma(x - \mu_1), \sigma^2) \cdot \phi(y; \mu_2^\bullet, \sigma^2) dy dx \\
&+ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(x; \mu_1, \sigma^2) \cdot \phi(x; \mu_1^\bullet, \sigma^2) \cdot \phi(y; 0, 1) \cdot \phi(y; 0, 1) dy dx.
\end{aligned}$$

I show how to find κ_1 and b_1 to bound the first summand in the last expression above. It is easy to similarly bound the second summand. By [Bromiley \(2003\)](#), the product of Gaussian densities $\phi(y; \mu_2 - \gamma(x - \mu_1), \sigma^2) \cdot \phi(y; \mu_2^\bullet, \sigma^2)$ is itself a Gaussian density in y , $\tilde{\phi}(y)$, multiplied by a scaling factor equal to $(4\pi\sigma^2)^{-1/2} \cdot \exp\left(-\frac{\gamma^2}{4\sigma^2} \cdot [x - (\mu_1 - \frac{\mu_2^\bullet}{\gamma} + \frac{\mu_2}{\gamma})]^2\right)$. So we have

$$\begin{aligned}
& \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(x; \mu_1, \sigma^2) \cdot \phi(x; \mu_1^\bullet, \sigma^2) \cdot \phi(y; \mu_2 - \gamma(x - \mu_1), \sigma^2) \cdot \phi(y; \mu_2^\bullet, \sigma^2) dy dx \\
&= \int_{-\infty}^{\infty} \phi(x; \mu_1, \sigma^2) \cdot \phi(x; \mu_1^\bullet, \sigma^2) \cdot (4\pi\sigma^2)^{-1/2} \cdot \exp\left(-\frac{\gamma^2}{4\sigma^2} \cdot [x - (\mu_1 - \frac{\mu_2^\bullet}{\gamma} + \frac{\mu_2}{\gamma})]^2\right) \cdot \int_{-\infty}^{\infty} \tilde{\phi}(y) dy dx \\
&= (4\pi\sigma^2)^{-1/2} \cdot \int_{-\infty}^{\infty} \phi(x; \mu_1, \sigma^2) \cdot \phi(x; \mu_1^\bullet, \sigma^2) \cdot \exp\left(-\frac{\gamma^2}{4\sigma^2} \cdot [x - (\mu_1 - \frac{\mu_2^\bullet}{\gamma} + \frac{\mu_2}{\gamma})]^2\right) \cdot dx.
\end{aligned}$$

Again applying [Bromiley \(2003\)](#), product of the two Gaussian densities $\phi(x; \mu_1, \sigma^2) \cdot \phi(x; \mu_1^\bullet, \sigma^2)$ is another Gaussian density with mean $\frac{\mu_1^\bullet + \mu_1}{2}$, variance $\frac{\sigma^2}{2}$, and multiplied to a scaling factor of $(4\pi\sigma^2)^{-1/2} \exp\left(-\frac{(\mu_1 - \mu_1^\bullet)^2}{4\sigma^2}\right)$. So above expression is:

$$K_1 \cdot \exp\left(-\frac{(\mu_1 - \mu_1^\bullet)^2}{4\sigma^2}\right) \cdot \int_{-\infty}^{\infty} \phi(x; \frac{\mu_1^\bullet + \mu_1}{2}, \frac{\sigma^2}{2}) \cdot \exp\left(-\frac{\gamma^2}{4\sigma^2} \cdot [x - (\mu_1 - \frac{\mu_2^\bullet}{\gamma} + \frac{\mu_2}{\gamma})]^2\right) dx$$

where K_1 is a constant not dependent on μ_1, μ_2 . Also, we may write

$$\exp\left(-\frac{\gamma^2}{4\sigma^2} \cdot [x - (\mu_1 - \frac{\mu_2^\bullet}{\gamma} + \frac{\mu_2}{\gamma})]^2\right) = K_2 \cdot \phi(x; (\mu_1 - \frac{\mu_2^\bullet}{\gamma} + \frac{\mu_2}{\gamma}), \sigma_B^2)$$

where $\sigma_B^2 = \frac{2\sigma^2}{\gamma^2}$ and $K_2 = (2\pi\sigma_B^2)^{1/2}$. Applying [Bromiley \(2003\)](#) one final time, the product $\phi(x; \frac{\mu_1^\bullet + \mu_1}{2}, \frac{\sigma^2}{2}) \cdot \phi(x; (\mu_1 - \frac{\mu_2^\bullet}{\gamma} + \frac{\mu_2}{\gamma}), \sigma_B^2)$ is a Gaussian density in x scaled by $K_4 \exp(-K_3 \cdot (\frac{\mu_1^\bullet + \mu_1}{2} - \frac{\mu_2 - \mu_2^\bullet}{\gamma})^2)$ where $K_3, K_4 > 0$ are constants not dependent on μ_1, μ_2 . So altogether, the second summand we are bounding is a constant multiple of $\exp\left(-\frac{(\mu_1 - \mu_1^\bullet)^2}{4\sigma^2}\right) \cdot \exp(-K_3 \cdot (\frac{\mu_1^\bullet + \mu_1}{2} - \frac{\mu_2 - \mu_2^\bullet}{\gamma})^2)$. For $|\mu_1| \geq |\mu_2|$, the max norm $\|(\mu_1, \mu_2)\| = |\mu_1|$ and $\exp\left(-\frac{(\mu_1 - \mu_1^\bullet)^2}{4\sigma^2}\right)$

decreases exponentially fast in the norm. For $|\mu_1| < |\mu_2|$, and $\frac{|\mu_2|}{2} - \frac{|\mu_1^\bullet|}{2} + \frac{|\mu_2^\bullet|}{\gamma} > 0$,

$$\exp(-K_3 \cdot (\frac{\mu_1^\bullet - \mu_1}{2} - \frac{\mu_2 - \mu_2^\bullet}{\gamma})^2) \leq \exp(-K_3 \cdot (\frac{|\mu_2|}{2} - \frac{|\mu_1^\bullet|}{2} + \frac{|\mu_2^\bullet|}{\gamma})^2).$$

So for large enough $|\mu_2|$, $\exp(-K_3 \cdot (\frac{\mu_1^\bullet - \mu_1}{2} - \frac{\mu_2 - \mu_2^\bullet}{\gamma})^2)$ will decrease exponentially fast in the norm. These two facts imply that there is some $K > 0$ so that whenever $\|(\mu_1, \mu_2)\| > K$,

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(x; \mu_1, \sigma^2) \cdot \phi(x; \mu_1^\bullet, \sigma^2) \cdot \phi(y; \mu_2 - \gamma(x - \mu_1), \sigma^2) \cdot \phi(y; \mu_2^\bullet, \sigma^2) dy dx < \|(\mu_1, \mu_2)\|^{-1}.$$

Now put $\kappa_1 = K^{-1}$ and we can ensure for any value of $\|(\mu_1, \mu_2)\|$ we will have

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(x; \mu_1, \sigma^2) \cdot \phi(x; \mu_1^\bullet, \sigma^2) \cdot \phi(y; \mu_2 - \gamma(x - \mu_1), \sigma^2) \cdot \phi(y; \mu_2^\bullet, \sigma^2) dy dx < \kappa_1 \cdot \|(\mu_1, \mu_2)\|^{-1}.$$

A5. There are positive constants b_2, b_3 so that for all (μ'_1, μ'_2) and $r > 0$ it holds that $g(S[(\mu'_1, \mu'_2), r]) \leq cr^{b_2}(1 + (\|(\mu'_1, \mu'_2)\| + r)^{b_3})$. Moreover, g assigns positive mass to every sphere with positive radius.

Since we have assumed that density g is bounded by B , the prior mass assigned to the sphere $S[(\mu'_1, \mu'_2), r]$ is bounded by B^2 times its Euclidean volume. So, take $b_2 = 2$ and $c = \pi B^2$ and the first statement is satisfied. Since we have assumed that g is strictly positive everywhere, the second statement is satisfied. \square

OA 1.2 Proof of Lemma A.16

Proof. Let $(\mu_1^\circ, \mu_2^\circ) \in \mathbb{R}^2$ be given. For any $\mu_1, \mu_2, c \in \mathbb{R}$, we have

$$\begin{aligned} U(c; \mu_1, \mu_2) &= \int_c^\infty u_1(x_1) \phi(x_1; \mu_1, \sigma^2) dx_1 \\ &\quad + \int_{-\infty}^c \left[\int_{-\infty}^\infty u_2(x_1, x_2) \phi(x_2; \mu_2 - \gamma(x_1 - \mu_1), \sigma^2) dx_2 \right] \cdot \phi(x_1; \mu_1, \sigma^2) dx_1. \end{aligned}$$

We first bound $|\int_c^\infty u_1(x_1) \phi(x_1; \mu_1, \sigma^2) dx_1 - \int_c^\infty u_1(x_1) \phi(x_1; \mu_1^\circ, \sigma^2) dx_1|$ by a multiple of $|\mu_1 - \mu_1^\circ|$. Suppose first $\mu_1 = \mu_1^\circ + \Delta$ for some $\Delta > 0$. We have

$$\int_c^\infty u_1(x_1) \phi(x_1; \mu_1, \sigma^2) dx_1 = \int_{c-\Delta}^\infty u_1(x_1 + \Delta) \phi(x_1; \mu_1^\circ, \sigma^2) dx_1.$$

By Lipschitz continuity of u_1 , $|u_1(x_1) - u_1(x_1 + \Delta)| \leq K_1 \Delta$ for all $x_1 \in \mathbb{R}$. Thus we conclude

$$\left| \int_c^\infty u_1(x_1) \phi(x_1; \mu_1, \sigma^2) dx_1 - \int_c^\infty u_1(x_1) \phi(x_1; \mu_1^\circ, \sigma^2) dx_1 \right| \leq K_1 \Delta + \left| \int_{c-\Delta}^c u_1(x_1) \phi(x_1; \mu_1^\circ, \sigma^2) dx_1 \right|.$$

Again by Lipschitz continuity of u_1 , for any $x_1 \in \mathbb{R}$,

$$|u_1(x_1)\phi(x_1; \mu_1, \sigma^2)| \leq (|u_1(0)| + K_1|x_1|) \cdot \phi(x_1; \mu_1^\circ, \sigma^2).$$

Since the Gaussian density decreases to 0 exponentially fast as $x_1 \rightarrow \pm\infty$, the RHS is uniformly bounded for all $x_1 \in \mathbb{R}$ by some constant, say $J_1 > 0$. (Note that the RHS is not a function of c , so J_1 does not depend on c .) This shows that

$$\left| \int_{c-\Delta}^c u_1(x_1)\phi(x_1; \mu_1^\circ, \sigma^2)dx_1 \right| \leq \int_{c-\Delta}^c |u_1(x_1)\phi(x_1; \mu_1^\circ, \sigma^2)|dx_1 \leq \int_{c-\Delta}^c J_1dx_1 = J_1\Delta.$$

So altogether,

$$\left| \int_c^\infty u_1(x_1)\phi(x_1; \mu_1, \sigma^2)dx_1 - \int_c^\infty u_1(x_1)\phi(x_1; \mu_1^\circ, \sigma^2)dx_1 \right| \leq (K_1 + J_1)\Delta.$$

If instead $\mu_1 = \mu_1^\circ - \Delta$, then a similar argument shows that

$$\left| \int_c^\infty u_1(x_1)\phi(x_1; \mu_1, \sigma^2)dx_1 - \int_c^\infty u_1(x_1)\phi(x_1; \mu_1^\circ, \sigma^2)dx_1 \right| \leq K_1\Delta + \left| \int_c^{c+\Delta} u_1(x_1)\phi(x_1; \mu_1^\circ, \sigma^2)dx_1 \right|,$$

and again we may bound the second term by $J_1\Delta$ as before.

We now turn to bounding the difference in the second summand making up $U(c; \mu_1, \mu_2)$. First consider the case where $\mu_2 = \mu_2^\circ$. For each $x_1 \in \mathbb{R}$, let $I(x_1; \mu_1) := \int_{-\infty}^\infty u_2(x_1, x_2)\phi(x_2; \mu_2^\circ - \gamma(x_1 - \mu_1), \sigma^2)dx_2$, the expected continuation utility after $X_1 = x_1$, in the subjective model $\Psi(\mu_1, \mu_2^\circ; \gamma)$. The second summand in $U(c; \mu_1, \mu_2)$ is given by $\int_{-\infty}^c I(x_1; \mu_1)\phi(x_1; \mu_1, \sigma^2)dx_1$.

For $x_1'' = x_1' + d_1$, $\mu_1'' = \mu_1' + d_2$, we have

$$\begin{aligned} I(x_1''; \mu_1'') &= \int_{-\infty}^\infty u_2(x_1'', x_2)\phi(x_2; \mu_2^\circ - \gamma(x_1'' - \mu_1''), \sigma^2)dx_2 \\ &= \int_{-\infty}^\infty u_2(x_1' + d_1, x_2 - \gamma(d_1 - d_2))\phi(x_2; \mu_2^\circ - \gamma(x_1' - \mu_1'), \sigma^2)dx_2. \end{aligned}$$

Lipschitz continuity of u_2 implies that

$$\begin{aligned} |u_2(x_1' + d_1, x_2 - \gamma(d_1 - d_2)) - u_2(x_1', x_2)| &\leq K_2((1 + \gamma) \cdot |d_1| + \gamma|d_2|) \\ &\leq K_2(1 + \gamma) \cdot (|d_1| + |d_2|), \end{aligned}$$

which shows $|I(x_1''; \mu_1'') - I(x_1'; \mu_1')| \leq K_2(1 + \gamma) \cdot (|x_1' - x_1''| + |x_2' - x_2'')$. That is, I is Lipschitz continuous.

Suppose $\mu_1 = \mu_1^\circ + \Delta$ for some $\Delta > 0$. Similar to the above argument bounding the first

summand in $(c; \mu_1, \mu_2)$, we have

$$\int_{-\infty}^c I(x_1; \mu_1) \phi(x_1; \mu_1, \sigma^2) dx_1 = \int_{-\infty}^{c-\Delta} I(x_1 + \Delta; \mu_1^\circ + \Delta) \phi(x_1; \mu_1^\circ, \sigma^2) dx_1.$$

By Lipschitz continuity of I , $|I(x_1; \mu_1^\circ) - I(x_1 + \Delta; \mu_1^\circ + \Delta)| \leq 2K_2(1 + \gamma)\Delta$ for all $x_1 \in \mathbb{R}$. Thus we conclude

$$\begin{aligned} & \left| \int_{-\infty}^c I(x_1; \mu_1) \phi(x_1; \mu_1, \sigma^2) dx_1 - \int_{-\infty}^c I(x_1; \mu_1^\circ) \phi(x_1; \mu_1^\circ, \sigma^2) dx_1 \right| \\ & \leq 2K_2(1 + \gamma)\Delta + \left| \int_{c-\Delta}^c I(x_1; \mu_1^\circ) \phi(x_1; \mu_1^\circ, \sigma^2) dx_1 \right|. \end{aligned}$$

Since $x_1 \mapsto I(x_1; \mu_1^\circ)$ is Lipschitz continuous, there exists $J_2 > 0$ so that $|I(x_1; \mu_1^\circ) \phi(x_1; \mu_1^\circ, \sigma^2)| \leq J_2$ for all $x_1 \in \mathbb{R}$, which means $|\int_{c-\Delta}^c I(x_1; \mu_1^\circ) \phi(x_1; \mu_1^\circ, \sigma^2) dx_1| \leq J_2\Delta$. (Once again, J_2 does not depend on c .) The case of $\mu_1 = \mu_1^\circ - \Delta$ is symmetric and we have shown that

$$\left| \int_{-\infty}^c I(x_1; \mu_1) \phi(x_1; \mu_1, \sigma^2) dx_1 - I(x_1; \mu_1^\circ) \phi(x_1; \mu_1^\circ, \sigma^2) dx_1 \right| \leq (2K_2(1 + \gamma) + J_2) \cdot |\mu_1 - \mu_1^\circ|.$$

Finally, we investigate the difference in the second summand of $U(c; \mu_1, \mu_2)$ between parameters (μ_1, μ_2°) and (μ_1, μ_2) for $\mu_1, \mu_2 \in \mathbb{R}$. This difference is bounded by

$$\int_{-\infty}^c \left| \int_{-\infty}^{\infty} u_2(x_1, x_2) \phi(x_2; \mu_2^\circ - \gamma(x_1 - \mu_1), \sigma^2) dx_2 - \int_{-\infty}^{\infty} u_2(x_1, x_2) \phi(x_2; \mu_2 - \gamma(x_1 - \mu_1), \sigma^2) dx_2 \right| \phi(x_1; \mu_1, \sigma^2) dx_1. \quad (3)$$

But for every $x_1 \in \mathbb{R}$,

$$\int_{-\infty}^{\infty} u_2(x_1, x_2) \phi(x_2; \mu_2 - \gamma(x_1 - \mu_1), \sigma^2) dx_2 = \int_{-\infty}^{\infty} u_2(x_1, x_2 + (\mu_2 - \mu_2^\circ)) \phi(x_2; \mu_2^\circ - \gamma(x_1 - \mu_1), \sigma^2) dx_2,$$

and $|u_2(x_1, x_2 + (\mu_2 - \mu_2^\circ)) - u_2(x_1, x_2)| \leq K_2|\mu_2 - \mu_2^\circ|$ by Lipschitz continuity of u_2 . This shows that, for all values $\mu_1, \mu_2 \in \mathbb{R}$, (3) is bounded by $K_2|\mu_2 - \mu_2^\circ|$.

Applying the triangle inequality to the second term, we conclude that

$$|U(c; \mu_1, \mu_2) - U(c; \mu_1^\circ, \mu_2^\circ)| \leq (K_1 + J_1)|\mu_1 - \mu_1^\circ| + (2K_2(1 + \gamma) + J_2) \cdot |\mu_1 - \mu_1^\circ| + K_2|\mu_2 - \mu_2^\circ|.$$

So we see that setting $K = K_1 + J_1 + (2K_2(1 + \gamma) + J_2)$ establishes the lemma. \square

OA 1.3 Proof of Lemma A.5

Proof. Consider the payoff difference between accepting x_1 and continuing under belief ν ,

$$D(x_1; \nu) := u_1(x_1) - \int \mathbb{E}_{X_2 \sim \mathcal{N}(\mu_2 - \gamma(x_1 - \mu_1^\bullet), \sigma^2)} [u_2(x_1, X_2)] d\nu(\mu_2).$$

Note that $D(x_1, \nu) = \int D(x_1; \mu_1^\bullet, \mu_2, \gamma) d\nu(\mu_2)$. Lemma A.2 shows that for every $\mu_2 \in \mathbb{R}$, $D(x_1; \mu_1^\bullet, \mu_2, \gamma)$ is strictly increasing in x_1 . Hence the same must hold for $D(x_1, \nu)$.

Also, Lemma A.2 implies there exists some $x'_1 \in \mathbb{R}$ so that $D(x'_1; \mu_1^\bullet, \underline{\mu}_2, \gamma) < 0$, and that there exists some $x''_1 \in \mathbb{R}$ satisfying $D(x''_1; \mu_1^\bullet, \underline{\mu}_2, \gamma) > 0$. Since u_2 increases in its second argument, we also get $D(x'_1; \mu_1^\bullet, \mu_2, \gamma) < 0$ and $D(x''_1; \mu_1^\bullet, \mu_2, \gamma) > 0$ for all $\mu_2 \in [\underline{\mu}_2, \bar{\mu}_2]$. This implies $D(x'_1; \nu) < 0$ and $D(x''_1; \nu) > 0$, as ν is supported on (a subset of) $[\underline{\mu}_2, \bar{\mu}_2]$.

Finally, I show $D(x_1; \nu)$ is continuous in x_1 . Fix $\bar{x}_1 \in \mathbb{R}$. Since u_1 is continuous, find $\delta > 0$ so that whenever $|x_1 - \bar{x}_1| < 1$, $|u_1(x_1) - u_1(\bar{x}_1)| < \delta$. Consider the function $f : \mathbb{R}^2 \rightarrow \mathbb{R}_{\geq 0}$ defined by $f(x_2, \mu_2) := |u_2(\bar{x}_1, x_2 - \gamma + \mu_2)| + |u_2(\bar{x}_1, x_2 + \gamma + \mu_2)| + \delta$.

Claim OA.1. Whenever $|x_1 - \bar{x}_1| < 1$, $|u_2(x_1, x_2 + \gamma(\bar{x}_1 - x_1) + \mu_2)| \leq f(x_2)$ for every $x_2, \mu_2 \in \mathbb{R}$.

Proof. This is the same as the proof of Claim A.1. □

Claim OA.2. $\int_{\underline{\mu}_2}^{\bar{\mu}_2} \left(\int_{-\infty}^{\infty} |f(x_2, \mu_2)| \cdot \phi(x_2; -\gamma(\bar{x}_1 - \mu_1^\bullet), \sigma^2) dx_2 \right) d\nu(\mu_2) < \infty$.

Proof. We may write

$$f(x_2, \mu_2) := u_{2,+}^{+\gamma}(x_2, \mu_2) + u_{2,-}^{+\gamma}(x_2, \mu_2) + u_{2,+}^{-\gamma}(x_2, \mu_2) + u_{2,-}^{-\gamma}(x_2, \mu_2) + \delta$$

where $u_{2,+}^{+\gamma}$ and $u_{2,-}^{+\gamma}$ are the positive and negative parts of $(x_2, \mu_2) \mapsto u_2(\bar{x}_1, x_2 + \gamma + \mu_2)$, and $u_{2,+}^{-\gamma}$ and $u_{2,-}^{-\gamma}$ are the positive and negative parts of $(x_2, \mu_2) \mapsto u_2(\bar{x}_1, x_2 - \gamma + \mu_2)$. From Assumption 1(d), for every $\mu_2 \in [\underline{\mu}_2, \bar{\mu}_2]$, each of $u_{2,+}^{+\gamma}(\cdot, \mu_2)$, $u_{2,-}^{+\gamma}(\cdot, \mu_2)$, $u_{2,+}^{-\gamma}(\cdot, \mu_2)$, and $u_{2,-}^{-\gamma}(\cdot, \mu_2)$ is integrable over \mathbb{R} with respect to the Gaussian density for $\mathcal{N}(-\gamma(\bar{x}_1 - \mu_1^\bullet), \sigma^2)$. These integrals are maximized at $\mu_2 = \bar{\mu}_2$ for $u_{2,+}^{+\gamma}(\cdot, \mu_2)$ and $u_{2,+}^{-\gamma}(\cdot, \mu_2)$, and maximized at $\mu_2 = \underline{\mu}_2$ for $u_{2,-}^{+\gamma}(\cdot, \mu_2)$ and $u_{2,-}^{-\gamma}(\cdot, \mu_2)$. In other words, for every $\mu_2 \in [\underline{\mu}_2, \bar{\mu}_2]$,

$$\begin{aligned} & \int_{-\infty}^{\infty} |f(x_2, \mu_2)| \cdot \phi(x_2; -\gamma(\bar{x}_1 - \mu_1^\bullet), \sigma^2) dx_2 \\ & \leq \int_{-\infty}^{\infty} \left(u_{2,+}^{+\gamma}(x_2, \bar{\mu}_2) + u_{2,-}^{-\gamma}(x_2, \bar{\mu}_2) \right) \cdot \phi(x_2; -\gamma(\bar{x}_1 - \mu_1^\bullet), \sigma^2) dx_2 \\ & \quad + \int_{-\infty}^{\infty} \left(u_{2,-}^{+\gamma}(x_2, \underline{\mu}_2) + u_{2,-}^{-\gamma}(x_2, \underline{\mu}_2) \right) \cdot \phi(x_2; -\gamma(\bar{x}_1 - \mu_1^\bullet), \sigma^2) dx_2. \end{aligned}$$

This bound is finite and does not depend on μ_2 , so the overall integral over $d\nu(\mu_2)$ is also finite. □

Consider a sequence $x_1^{(n)} \rightarrow \bar{x}_1$. We have

$$\begin{aligned} D(x_1^{(n)}; \nu) &= u_1(x_1^{(n)}) - \int \mathbb{E}_{\tilde{X}_2 \sim \mathcal{N}(\mu_2 - \gamma(x_1^{(n)} - \mu_1^\bullet), \sigma^2)} [u_2(x_1^{(n)}, \tilde{X})] d\nu(\mu_2) \\ &= u_1(x_1^{(n)}) - \int \mathbb{E}_{\tilde{X}_2 \sim \mathcal{N}(-\gamma(\bar{x}_1 - \mu_1^\bullet), \sigma^2)} [u_2(x_1^{(n)}, \tilde{X}_2 + \gamma(\bar{x}_1 - x_1^{(n)}) + \mu_2)] d\nu(\mu_2) \\ &= u_1(x_1^{(n)}) - \int_{\mu_2}^{\bar{\mu}_2} \int_{-\infty}^{\infty} u_2(x_1^{(n)}, x_2 + \gamma(\bar{x}_1 - x_1^{(n)}) + \mu_2) \cdot \phi(x_2; -\gamma(\bar{x}_1 - \mu_1^\bullet), \sigma^2) dx_2 d\nu(\mu_2). \end{aligned}$$

The sequence of functions $(x_2, \mu_2) \mapsto u_2(x_1^{(n)}, x_2 + \gamma(\bar{x}_1 - x_1^{(n)}) + \mu_2)$ pointwise converge to $u_2(\bar{x}_1, x_2 + \mu_2)$ as $n \rightarrow \infty$. From the two claims, for all large enough n , this sequence of functions are pointwise dominated by f , an absolutely integrable function on the same domain. Therefore continuity follows from dominated convergence theorem, as in the proof of Lemma A.2.

This means there exists a unique c^* so that $D(c^*) = 0$. The cutoff strategy S_{c^*} is optimal, because it stops at every x_1 whose stopping payoff exceeds expected continuation payoff, and continues at every x_1 where expected continuation payoff is higher than stopping payoff.

For any $c' = c^* + \delta$ for some $\delta > 0$, the difference in expected payoffs of S_{c^*} and $S_{c'}$ is $\int_{c^*}^{c^* + \delta} D(x_1; \nu) > 0$ since $D(x_1; \nu)$ is strictly positive on the interval $(c^*, c^* + \delta]$. So every strictly higher cutoff than c^* is strictly suboptimal. A similar argument shows every strictly lower cutoff than c^* is also strictly suboptimal. \square

OA 1.4 Proof of Lemma A.18

Proof. For Assumption A.1(a), the marginal of $F(\cdot; \theta_1, \theta_2)$ on X_1 is simply $Q_1(\cdot; \theta_1)$, which I assumed is strictly increasing in mean with respect to θ_1 .

For Assumptions A.1(b), it is well-known that by the copula construction, for all $u, v \in [0, 1]$ $\mathbb{P}_{F(\cdot; \theta_1, \theta_2)}[X_2 \leq Q_2^{-1}(v; \theta_2) \mid X_1 = Q_1^{-1}(u; \theta_1)] = \frac{\partial W}{\partial u}(u, v)$. This means $\frac{\partial W}{\partial u}(u, v)$ is increasing in v . Fixing some $x_1 \in I_1$ and $\theta_1 \in \Theta_1$, put $u = Q_1(x_1; \theta_1)$. Now for every θ_2 and $x_2 \in I_2$, we have $\mathbb{P}_{F(\cdot; \theta_1, \theta_2)}[X_2 \leq x_2 \mid X_1 = x_1] = \frac{\partial W}{\partial u}(u, Q_2^{-1}(x_2; \theta_2))$. Since the family of marginals $Q_2(\cdot; \theta_2)$ increases in FOSD order as θ_2 increases, $Q_2^{-1}(x_2; \theta_2)$ is decreasing in θ_2 . Since $\frac{\partial W}{\partial u}$ increases in its second argument, $\mathbb{P}_{F(\cdot; \theta_1, \theta_2)}[X_2 \leq x_2 \mid X_1 = x_1]$ must then decrease in θ_2 , that is to say the conditional distribution $X_2 \mid X_1 = x_1$ is increasing in FOSD order in θ_2 . So in particular Assumption A.1(b) is satisfied.

For Assumption A.1(c), again start with the expression

$$\mathbb{P}_{F(\cdot; \theta_1, \theta_2)}[X_2 \leq Q_2^{-1}(v; \theta_2) \mid X_1 = Q_1^{-1}(u; \theta_1)] = \frac{\partial W}{\partial u}(u, v).$$

For $x_1'' > x_1'$, put $u'' = Q_1(x_1'') > Q_1(x_1') = u'$. We have for every $v \in [0, 1]$ that

$$\mathbb{P}_{F(\cdot; \theta_1, \theta_2)}[X_2 \leq Q_2^{-1}(v; \theta_2) \mid X_1 = x_1'] = \frac{\partial W}{\partial u}(Q_1(x_1'; \theta_1), v)$$

while

$$\mathbb{P}_{F(\cdot; \theta_1, \theta_2)}[X_2 \leq Q_2^{-1}(v; \theta_2) \mid X_1 = x_1''] = \frac{\partial W}{\partial u}(Q_1(x_1''; \theta_1), v).$$

Since the distribution function $Q_1(\cdot; \theta_1)$ has full support, $Q_1(x_1''; \theta_1) > Q_1(x_1'; \theta_1)$. And since we assumed $\frac{\partial W}{\partial u}$ is increasing in its first argument, we see that $\mathbb{P}_{F(\cdot; \theta_1, \theta_2)}[X_2 \leq x_2 \mid X_1 = x_1]$ is increasing in x_1 . That is, the conditional distribution $X_2 \mid X_1 = x_2$ is decreasing in FOSD order in x_1 . So Assumption [A.1\(c\)](#) is satisfied. \square

OA 1.5 Proof of Lemma [A.19](#)

Proof. Suppose (θ_1^M, θ_2^M) is an MOM estimator. I show any other MOM estimator $(\hat{\theta}_1, \hat{\theta}_2)$ must be equal to it.

We may rewrite the moments as: $m_1[\mathcal{H}(\theta_1^M, \theta_2^M; c)] = \mathbb{E}_{F_1(\cdot; \theta_1)}[X_1]$, $m_2[\mathcal{H}(\theta_1^M, \theta_2^M; c)] = \mathbb{E}_{F(\cdot; \theta_1, \theta_2)}[X_2 \mid X_1 < c]$.

The unconditional mean of X_1 , namely $\mathbb{E}_{F_1(\cdot; \theta_1)}[X_1]$, is strictly increasing in θ_1 by Assumption [A.1\(a\)](#). So, at most one value of $\theta_1 \in \Theta_1$ can generate an unconditional mean that matches $m_1[\mathcal{H}^\bullet(c)]$, meaning we must have $\hat{\theta}_1 = \theta_1^M$.

Given this unique θ_1^M , Assumption [A.1\(b\)](#) implies the conditional mean $\mathbb{E}_{F_{2|1}(\cdot; \theta_1^M, \theta_2 | x_1)}[X_2]$ is strictly increasing in θ_2 for every $x_1 < c$. The conditional mean $\mathbb{E}_{F(\cdot; \theta_1^M, \theta_2)}[X_2 \mid X_1 < c]$ is given by an integral over $\mathbb{E}_{F_{2|1}(\cdot; \theta_1^M, \theta_2 | x_1)}[X_2]$ across the values $x_1 < c$, therefore $\mathbb{E}_{F(\cdot; \theta_1^M, \theta_2)}[X_2 \mid X_1 < c]$ is also strictly increasing in θ_2 . So there is at most one value of θ_2 such that $m_2[\mathcal{H}(\theta_1^M, \theta_2; c)] = m_2[\mathcal{H}^\bullet(c)]$, which gives $\hat{\theta}_2 = \theta_2^M$. \square

OA 1.6 Proof of Proposition [A.4](#)

Proof. Since the marginal distribution of X_1 in $F(\cdot; \theta_1, \theta_2)$ only depends on θ_1 and is strictly increasing in it, and since $m_1[\mathcal{H}^\bullet(c)]$ does not depend on c , we must have $\theta_1^M(c') = \theta_1^M(c'')$. I denote this common value by θ_1^M .

In seeking to match the moment $m_2[\mathcal{H}^\bullet(c'')]$, we may break down the conditioning event

$\{X_1 < c''\}$ into the union $\{X_1 < c'\} \cup \{c' \leq X_1 < c''\}$, so

$$\begin{aligned} \mathbb{E}_{F(\cdot; \theta_1^M, \theta_2^M(c''))}[X_2 | X_1 < c''] &= \frac{\mathbb{P}_{F(\cdot; \theta_1^M, \theta_2^M(c''))}[X_1 < c']}{\mathbb{P}_{F(\cdot; \theta_1^M, \theta_2^M(c''))}[X_1 < c'']} \cdot \mathbb{E}_{F(\cdot; \theta_1^M, \theta_2^M(c''))}[X_2 | X_1 < c'] \\ &+ \frac{\mathbb{P}_{F(\cdot; \theta_1^M, \theta_2^M(c''))}[c' \leq X_1 < c'']}{\mathbb{P}_{F(\cdot; \theta_1^M, \theta_2^M(c''))}[X_1 < c'']} \cdot \mathbb{E}_{F(\cdot; \theta_1^M, \theta_2^M(c''))}[X_2 | c' \leq X_1 < c'']. \end{aligned}$$

Suppose by way of contradiction that $\theta_2^M(c'') \leq \theta_2^M(c')$. Then since $\mathbb{E}_{F_{2|1}(\cdot; \theta_1^M, \theta_2 | x_1)}[X_2]$ is strictly increasing in θ_2 for every x_1 by Assumption A.1(b), we get

$$\mathbb{E}_{F(\cdot; \theta_1^M, \theta_2^M(c''))}[X_2 | X_1 < c'] \leq \mathbb{E}_{F(\cdot; \theta_1^M, \theta_2^M(c'))}[X_2 | X_1 < c'] = m_2[\mathcal{H}^\bullet(c')],$$

where the equality comes from the second moment condition of the MOM estimator $(\theta_1^M(c'), \theta_2^M(c'))$. Similarly

$$\mathbb{E}_{F(\cdot; \theta_1^M, \theta_2^M(c''))}[X_2 | c' \leq X_1 < c''] \leq \mathbb{E}_{F(\cdot; \theta_1^M, \theta_2^M(c'))}[X_2 | c' \leq X_1 < c''].$$

Now $\mathbb{E}_{F_{2|1}(\cdot; \theta_1^M, \theta_2^M(c') | x_1)}[X_2]$ is strictly decreasing in x_1 by Assumption A.1(c), so for every $x_1 \in [c', c'')$ the expectation is smaller than for every $x_1 < c'$. This shows that

$$\mathbb{E}_{F(\cdot; \theta_1^M, \theta_2^M(c'))}[X_2 | c' \leq X_1 < c''] < \mathbb{E}_{F(\cdot; \theta_1^M, \theta_2^M(c'))}[X_2 | X_1 < c'] = m_2[\mathcal{H}^\bullet(c')].$$

Since $F(\cdot; \theta_1^M, \theta_2^M(c''))$ has full support on $I_1 \times I_2$, the probability $\mathbb{P}_{F(\cdot; \theta_1^M, \theta_2^M(c''))}[c' \leq X_1 < c'']$ is strictly positive since $[c', c'')$ is an interval in the interior of I_1 . So we see that $\mathbb{E}_{F(\cdot; \theta_1^M, \theta_2^M(c''))}[X_2 | X_1 < c'']$ is a convex combination between a term that is no larger than $m_2[\mathcal{H}^\bullet(c'')]$ and another term that is strictly smaller than $m_2[\mathcal{H}^\bullet(c')]$, with strictly positive weight on the latter. Since $m_2[\mathcal{H}^\bullet(c')] = m_2[\mathcal{H}^\bullet(c'')]$, we see that $(\theta_1^M(c''), \theta_2^M(c''))$ cannot match the second moment condition of $m_2[\mathcal{H}(\theta_1^M(c''), \theta_2^M(c''); c'')] = m_2[\mathcal{H}^\bullet(c'')]$, contradiction. Hence we conclude $\theta_2^M(c'') > \theta_2^M(c')$. \square

OA 1.7 Proof of Corollary A.1

Proof. I first show that under any of the models $F(\cdot; \theta_1, \theta_2)$, agent's subjectively optimal stopping rule is a cutoff rule (possibly involving never stopping or always stopping). It suffices to show that

$$x_1 \mapsto (u_1(x_1) - \mathbb{E}_{F_{2|1}(\cdot; \theta_1, \theta_2 | x_1)}[u_2(x_1, X_2)])$$

is strictly increasing in x_1 . By linearity of u_2 in its second argument, this expression is equal to

$$x_1 \mapsto (u_1(x_1) - u_2(x_1, \mathbb{E}_{F_{2|1}(\cdot; \theta_1, \theta_2 | x_1)})).$$

Suppose $x'_1 > x''_1$. By Assumption 1(b),

$$u_1(x'_1) - u_2(x'_1, \mathbb{E}_{F_{2|1}(\cdot; \theta_1, \theta_2 | x'_1)}) \geq u_1(x''_1) - u_2(x''_1, \mathbb{E}_{F_{2|1}(\cdot; \theta_1, \theta_2 | x'_1)}).$$

By Assumption A.1(c), $\mathbb{E}_{F_{2|1}(\cdot; \theta_1, \theta_2 | x'_1)} < \mathbb{E}_{F_{2|1}(\cdot; \theta_1, \theta_2 | x''_1)}$. Combined with Assumption 1(a), it gives $u_2(x''_1, \mathbb{E}_{F_{2|1}(\cdot; \theta_1, \theta_2 | x'_1)}) < u_2(x''_1, \mathbb{E}_{F_{2|1}(\cdot; \theta_1, \theta_2 | x''_1)})$, hence showing

$$u_1(x'_1) - u_2(x'_1, \mathbb{E}_{F_{2|1}(\cdot; \theta_1, \theta_2 | x'_1)}) > u_1(x''_1) - u_2(x''_1, \mathbb{E}_{F_{2|1}(\cdot; \theta_1, \theta_2 | x''_1)}).$$

Also, suppose $F(\cdot; \theta_1, \theta'_2)$ induces either a stopping threshold which is an interior point of I_1 , or always stopping. Then $F(\cdot; \theta_1, \theta''_2)$ induces a higher stopping threshold or always stopping whenever $\theta'_2 \geq \theta''_2$. To see this, if there is an indifference point \bar{x}_1 in the interior of I_1 with $u_1(\bar{x}_1) = u_2(\bar{x}_1, \mathbb{E}_{F_{2|1}(\cdot; \theta_1, \theta''_2 | \bar{x}_1)})$, then we have $\mathbb{E}_{F_{2|1}(\cdot; \theta_1, \theta'_2 | \bar{x}_1)} > \mathbb{E}_{F_{2|1}(\cdot; \theta_1, \theta''_2 | \bar{x}_1)}$ due to Assumption A.1(b), so $u_1(\bar{x}_1) < u_2(\bar{x}_1, \mathbb{E}_{F_{2|1}(\cdot; \theta_1, \theta'_2 | \bar{x}_1)})$. This shows under $F(\cdot; \theta_1, \theta'_2)$ the agent strictly prefers continuing at \bar{x}_1 , so the acceptance threshold must be higher. Similarly, if the agent prefers always stopping at every $x_1 \in I_1$ under $F(\cdot; \theta_1, \theta''_2)$. then she prefers strictly stopping at every x_1 under $F(\cdot; \theta_1, \theta'_2)$.

I now show that $\mu_{1,[t]}^M$, $\mu_{2,[t]}^M$, and $c_{[t]}^M$ are well defined for every $t \geq 1$. MOM agents in generation $t \geq 1$ face t sub-datasets of censored histories, with the distribution $\mathcal{H}^\bullet(c_{[0]}), \dots, \mathcal{H}^\bullet(c_{[t-1]})$ where $c_{[0]} \in \text{int}(I_1)$. The moments to match are

$$m_1(\mathcal{H}^\bullet(c_{[0]}), \dots, \mathcal{H}^\bullet(c_{[K-1]})) = \mathbb{E}_{F^\bullet}[X_1],$$

$$m_2(\mathcal{H}^\bullet(c_{[0]}), \dots, \mathcal{H}^\bullet(c_{[K-1]})) = \mathbb{E}_{F^\bullet}[X_2],$$

where the second-period moment is well defined because $c_{[0]}$ is interior, so a positive fraction of histories in at least one sub-dataset contain uncensored X_2 . These moments are interior values in I_1, I_2 respectively, since F^\bullet has full-support marginal distributions.

By Assumption A.2(b), there exists $\bar{\theta}_1 \in \Theta_1$, independent of K and $(c_{[0]}, \dots, c_{[K-1]})$, so that $\mathbb{E}_{F_1(\cdot; \bar{\theta}_1)}[X_1] = \mathbb{E}_{F^\bullet}[X_1]$. By combining Assumption A.1(b) and A.2(c), we get that

$$\theta_2 \mapsto m_2(\mathcal{H}(\bar{\theta}_1, \theta_2; c_{[0]}), \dots, \mathcal{H}(\bar{\theta}_1, \theta_2; c_{[K-1]}))$$

is increasing, continuous on Θ_2 with a range of I_2 . (This uses the fact that $c_{[0]}$ is in the interior

of I_1 .) Since MOM agents are matching an interior value $\mathbb{E}_{F^\bullet}[X_2] \in \text{int}(I_2)$, this shows that for any K and $(c_{[0]}, \dots, c_{[K-1]})$ with $c_{[0]} \in \text{int}(I_1)$, $\theta_1^M(c_{[0]}, \dots, c_{[K-1]})$ and $\theta_2^M(c_{[0]}, \dots, c_{[K-1]})$ exist, and furthermore $\theta_1^M(c_{[0]}, \dots, c_{[K-1]}) = \bar{\theta}_1$.

By uniqueness of MOM estimators in Lemma A.19, $\mu_{1,[t]}^M, \mu_{2,[t]}^M$ are well defined for each $t \geq 1$. Also, $c_{[t]}^M$ is also well defined for each $t \geq 1$, given that we have shown the optimal strategy in the model $F(\cdot; \mu_{1,[t]}^M, \mu_{2,[t]}^M)$ is a cutoff strategy.

To prove monotonicity, first suppose that $c_{[1]} \leq c_{[0]}$. I have argued that we must have $\theta_{1,[2]}^M = \theta_{1,[1]}^M = \bar{\theta}_1$, so now I rule out $\theta_{2,[2]}^M > \theta_{2,[1]}^M$. Note that

$$m_2(\mathcal{H}(\bar{\theta}_1, \theta_2; c_{[0]}), \mathcal{H}(\bar{\theta}_1, \theta_2; c_{[1]})) = \frac{w_0}{w_0 + w_1} m_2(\mathcal{H}(\bar{\theta}_1, \theta_2; c_{[0]})) + \frac{w_1}{w_0 + w_1} m_2(\mathcal{H}(\bar{\theta}_1, \theta_2; c_{[1]}))$$

where $w_0 = \mathbb{P}_{F^\bullet}[X_1 \leq c_{[0]}] > 0$ and $w_1 = \mathbb{P}_{F^\bullet}[X_1 \leq c_{[1]}] \geq 0$. The moment-matching condition for generation 1 implies $m_2(\mathcal{H}(\bar{\theta}_1, \theta_{2,[1]}^M; c_{[0]})) = \mathbb{E}_{F^\bullet}[X_2]$. For any $\theta_{2,[2]}^M > \theta_{2,[1]}^M$, we have

$$m_2(\mathcal{H}(\bar{\theta}_1, \theta_{2,[2]}^M; c_{[0]})) > \mathbb{E}_{F^\bullet}[X_2]$$

from Assumption A.2(b). If $c_{[1]} = \inf(I_1)$, we have found a contradiction since the weight w_1 is 0. When $c_{[1]} > \inf(I_1)$, we get

$$m_2(\mathcal{H}(\bar{\theta}_1, \theta_{2,[2]}^M; c_{[1]})) \geq m_2(\mathcal{H}(\bar{\theta}_1, \theta_{2,[2]}^M; c_{[0]})) > \mathbb{E}_{F^\bullet}[X_2]$$

by combining Assumption A.2(c) with the fact that $c_{[1]} \leq c_{[0]}$. Both w_0 and w_1 are strictly positive, and they are multiplied to terms both strictly larger than $\mathbb{E}_{F^\bullet}[X_2]$. This shows

$$m_2(\mathcal{H}(\bar{\theta}_1, \theta_{2,[2]}^M; c_{[0]}), \mathcal{H}(\bar{\theta}_1, \theta_{2,[2]}^M; c_{[1]})) > \mathbb{E}_{F^\bullet}[X_2],$$

again contradicting the moment condition.

Hence we must have $\theta_{2,[2]}^M \leq \theta_{2,[1]}^M$, and thus $c_{[2]}^M \leq c_{[1]}^M$ by monotonicity of the cutoff threshold in belief as discuss before. Similar argument establishes $(\mu_{1,[t]}^M)_{t \geq 1}$, $(\mu_{2,[t]}^M)_{t \geq 1}$, and $(c_{[t]}^M)_{t \geq 1}$ are decreasing sequences.

The case of $c_{[1]} > c_{[0]}$ is symmetric. □

OA 1.8 Proof of Proposition A.6

Let

$$J_1 = \frac{(\mu_1^\bullet - \mu_1)^2}{2\sigma^2}$$

and for $i \geq 2$, let J_i be

$$\int_{-\infty}^{c_1} \dots \int_{-\infty}^{c_{i-1}(x_1, \dots, x_{i-2})} \prod_{j=1}^{i-1} \phi(x_j; \mu_j^\bullet, \sigma^2) \cdot \left[\frac{(\mu_i^\bullet - \mu_i + \sum_{j=1}^{i-1} \gamma_{i,j} \cdot (x_j - \mu_j))^2}{2\sigma^2} \right] dx_{i-1} \dots dx_1.$$

The expression in square brackets is the KL divergence from the agent's subjective model for $X_i | (X_1 = x_1, \dots, X_{i-1} = x_{i-1})$ to the true distribution of X_i , under fundamentals μ_1, \dots, μ_i . So, the integral J_i is a weighted average of this divergence, taken across different realizations of previous draws (x_1, \dots, x_{i-1}) with weights given by the true likelihood of observing such a sequence of draws in periods 1 through $i-1$ under the stopping strategy S_c . Note that for each i , J_i (and I_i) depends on μ_1, \dots, μ_i .

I first develop an alternative expression of $D_{KL}(\mathcal{H}(\Psi^\bullet; S_c) || \mathcal{H}(\Psi(\mu; \gamma); S_c))$ as the sum of J_i .

Lemma OA.1. $\sum_{i=1}^L I_i = \sum_{i=1}^L J_i$.

Proof. Let \tilde{I}_i be a slightly modified version of I_i , where the inner-most integral over x_i has the range $(-\infty, \infty)$, so \tilde{I}_i is

$$\int_{-\infty}^{c_1} \dots \int_{-\infty}^{c_{i-1}(x_1, \dots, x_{i-2})} \int_{-\infty}^{\infty} \prod_{k=1}^i \phi(x_k; \mu_k^\bullet, \sigma^2) \ln \left(\frac{\prod_{k=1}^i \phi(x_k; \mu_k^\bullet, \sigma^2)}{\prod_{k=1}^i \phi(x_k; \mu_k - \sum_{j=1}^{k-1} \gamma_{k,j} \cdot (x_j - \mu_j), \sigma^2)} \right) dx_i \dots dx_1.$$

Observe that $\tilde{I}_L = I_L$. Inductively I will show $\tilde{I}_{L'} + \sum_{i=1}^{L'-1} I_i = \sum_{i=1}^{L'} J_i$ for every $1 \leq L' \leq L$. When $L' = 1$, this just says $\tilde{I}_1 = J_1$, which is true by definition. Now suppose the statement holds for some $L' = S \leq L-1$. I show it also holds when $L' = S+1$.

We have

$$\tilde{I}_{S+1} + \sum_{i=1}^S I_i = \tilde{I}_{S+1} + (I_S - \tilde{I}_S) + \left(\tilde{I}_S + \sum_{i=1}^{S-1} I_i \right) = \tilde{I}_{S+1} + (I_S - \tilde{I}_S) + \sum_{i=1}^S J_i$$

where the last equality comes from the inductive hypothesis. Since I_S and \tilde{I}_S simply differ in terms of the bounds of the inner-most integral, $I_S - \tilde{I}_S$ is

$$- \int_{-\infty}^{c_1} \dots \int_{-\infty}^{c_S(x_1, \dots, x_{S-1})} \prod_{k=1}^S \phi(x_k; \mu_k^\bullet, \sigma^2) \cdot \ln \left(\frac{\prod_{k=1}^S \phi(x_k; \mu_k^\bullet, \sigma^2)}{\prod_{k=1}^S \phi(x_k; \mu_k - \sum_{j=1}^{k-1} \gamma_{k,j} \cdot (x_j - \mu_j), \sigma^2)} \right) dx_S \dots dx_1.$$

Now, decompose the $\ln \left(\frac{\prod_{k=1}^{S+1} \phi(x_k; \mu_k^\bullet, \sigma^2)}{\prod_{k=1}^{S+1} \phi(x_k; \mu_k - \sum_{j=1}^{k-1} \gamma_{k,j} \cdot (x_j - \mu_j), \sigma^2)} \right)$ term in the integrand of \tilde{I}_{S+1} into the sum

$$\ln \left(\frac{\prod_{k=1}^S \phi(x_k; \mu_k^\bullet, \sigma^2)}{\prod_{k=1}^S \phi(x_k; \mu_k - \sum_{j=1}^{k-1} \gamma_{k,j} \cdot (x_j - \mu_j), \sigma^2)} \right) + \ln \left(\frac{\phi(x_{S+1}; \mu_{S+1, \bullet}, \sigma^2)}{\phi(x_{S+1}; \mu_{S+1} - \sum_{j=1}^S \gamma_{S+1,j} \cdot (x_j - \mu_j), \sigma^2)} \right).$$

We know that

$$\begin{aligned}
& \int_{-\infty}^{c_1} \dots \int_{-\infty}^{c_S(\dots)} \int_{-\infty}^{\infty} \prod_{k=1}^{S+1} \phi(x_k; \mu_k^\bullet, \sigma^2) \ln \left(\frac{\prod_{k=1}^S \phi(x_k; \mu_k^\bullet, \sigma^2)}{\prod_{k=1}^S \phi(x_k; \mu_k - \sum_{j=1}^{k-1} \gamma_{k,j} \cdot (x_j - \mu_j), \sigma^2)} \right) dx_{S+1} \dots dx_1 \\
&= \int_{-\infty}^{c_1} \dots \int_{-\infty}^{c_S(\dots)} \prod_{k=1}^S \phi(x_k; \mu_k^\bullet, \sigma^2) \int_{-\infty}^{\infty} \phi(x_{S+1}; \mu_{S+1}^\bullet, \sigma^2) \cdot \ln \left(\frac{\prod_{k=1}^S \phi(x_k; \mu_k^\bullet, \sigma^2)}{\prod_{k=1}^S \phi(x_k; \mu_k - \sum_{j=1}^{k-1} \gamma_{k,j} \cdot (x_j - \mu_j), \sigma^2)} \right) dx_{S+1} \dots dx_1 \\
&= \int_{-\infty}^{c_1} \dots \int_{-\infty}^{c_S(\dots)} \prod_{k=1}^S \phi(x_k; \mu_k^\bullet, \sigma^2) \cdot \ln \left(\frac{\prod_{k=1}^S \phi(x_k; \mu_k^\bullet, \sigma^2)}{\prod_{k=1}^S \phi(x_k; \mu_k - \sum_{j=1}^{k-1} \gamma_{k,j} \cdot (x_j - \mu_j), \sigma^2)} \right) dx_S \dots dx_1 \\
&= -(I_S - \tilde{I}_S)
\end{aligned}$$

where $c_S(\dots)$ abbreviates the bound of integration $c_S(x_1, \dots, x_{S-1})$. At the same time,

$$\begin{aligned}
& \int_{-\infty}^{c_1} \dots \int_{-\infty}^{c_S(\dots)} \int_{-\infty}^{\infty} \prod_{k=1}^{S+1} \phi(x_k; \mu_k^\bullet, \sigma^2) \ln \left(\frac{\phi(x_{S+1}; \mu_{S+1}^\bullet, \sigma^2)}{\phi(x_{S+1}; \mu_{S+1} - \sum_{j=1}^S \gamma_{S+1,j} \cdot (x_j - \mu_j), \sigma^2)} \right) dx_{S+1} \dots dx_1 \\
&= \int_{-\infty}^{c_1} \dots \int_{-\infty}^{c_S(\dots)} \prod_{k=1}^S \phi(x_k; \mu_k^\bullet, \sigma^2) \int_{-\infty}^{\infty} \phi(x_{S+1}; \mu_{S+1}^\bullet, \sigma^2) \ln \left(\frac{\phi(x_{S+1}; \mu_{S+1}^\bullet, \sigma^2)}{\phi(x_{S+1}; \mu_{S+1} - \sum_{j=1}^S \gamma_{S+1,j} \cdot (x_j - \mu_j), \sigma^2)} \right) dx_{S+1} \dots dx_1 \\
&= \int_{-\infty}^{c_1} \dots \int_{-\infty}^{c_S(\dots)} \prod_{k=1}^S \phi(x_k; \mu_k^\bullet, \sigma^2) D_{KL}[\mathcal{N}(\mu_{S+1}^\bullet, \sigma^2), \mathcal{N}(\mu_{S+1} - \sum_{j=1}^S \gamma_{S+1,j} \cdot (x_j - \mu_j), \sigma^2)] dx_S \dots dx_1 \\
&= \int_{-\infty}^{c_1} \dots \int_{-\infty}^{c_S(\dots)} \prod_{k=1}^S \phi(x_k; \mu_k^\bullet, \sigma^2) \frac{(\mu_{S+1}^\bullet - \mu_{S+1} + \sum_{j=1}^S \gamma_{S+1,j} \cdot (x_j - \mu_j))^2}{2\sigma^2} dx_S \dots dx_1 \\
&= J_{S+1}
\end{aligned}$$

where we used the closed-form expression of the KL divergence between two Gaussian distributions,

$$D_{KL} \left[\mathcal{N}(\mu_{S+1}^\bullet, \sigma^2) \parallel \mathcal{N}(\mu_{S+1} - \sum_{j=1}^S \gamma_{S+1,j} \cdot (x_j - \mu_j), \sigma^2) \right] = \frac{(\mu_{S+1}^\bullet - \mu_{S+1} + \sum_{j=1}^S \gamma_{S+1,j} \cdot (x_j - \mu_j))^2}{2\sigma^2}.$$

So by induction, $\tilde{I}_L + \sum_{i=1}^{L-1} I_i = \sum_{i=1}^L J_i$. As $\tilde{I}_L = I_L$, we are done. \square

Using Lemma OA.1, I can now give the proof of Proposition A.6.

Proof. Abbreviate $D_{KL}(\mathcal{H}(\Psi^\bullet; S_c) \parallel \mathcal{H}(\Psi(\boldsymbol{\mu}; \boldsymbol{\gamma}); S_c))$ as $\xi(\mu_1, \dots, \mu_L)$. By Lemma OA.1, $\xi(\mu_1, \dots, \mu_L) = \sum_{i=1}^L J_i(\mu_1, \dots, \mu_i)$. We show that the recursively defined parameters are the only ones satisfying the first-order condition, $\frac{\partial \xi}{\partial \mu_i}(\hat{\mu}_1, \dots, \hat{\mu}_L) = 0$ for each i .

In the integrand for J_i , each μ_j where $1 \leq j \leq i$ appears once in the term $\frac{(\mu_i^\bullet - \mu_i + \sum_{j=1}^{i-1} \gamma_{i,j} \cdot (x_j - \mu_j))^2}{2\sigma^2}$. For any (x_1, \dots, x_{i-1}) , the partial derivative of this term with respect to μ_j for $j < i$ is $\gamma_{i,j}$ times its partial derivative with respect to μ_i . That is, at any values of $\hat{\mu}_1, \dots, \hat{\mu}_i$, we get

$$\frac{\partial J_i}{\partial \mu_j}(\hat{\mu}_1, \dots, \hat{\mu}_i) = \gamma_{i,j} \frac{\partial J_i}{\partial \mu_i}(\hat{\mu}_1, \dots, \hat{\mu}_i)$$

for each $1 \leq j < i$.

At any $(\mu_1^*, \dots, \mu_L^*)$ satisfying the first-order condition for μ_L , we must have

$$\frac{\partial \xi}{\partial \mu_L}(\mu_1^*, \dots, \mu_L^*) = \frac{\partial J_L}{\partial \mu_L}(\mu_1^*, \dots, \mu_L^*) = 0.$$

By above, this also implies for each $1 \leq j < L$, either $\frac{\partial J_L}{\partial \mu_j}(\mu_1^*, \dots, \mu_L^*) = 0$, or $\gamma_{L,k} = 0$ (in which case J_L is not actually a function of μ_j and $\frac{\partial J_L}{\partial \mu_j} = 0$ everywhere). Either way, this shows for the case of $j = L - 1$,

$$\begin{aligned} \frac{\partial \xi}{\partial \mu_{L-1}}(\mu_1^*, \dots, \mu_L^*) &= \frac{\partial J_L}{\partial \mu_{L-1}}(\mu_1^*, \dots, \mu_L^*) + \frac{\partial J_{L-1}}{\partial \mu_{L-1}}(\mu_1^*, \dots, \mu_{L-1}^*) \\ &= \frac{\partial J_{L-1}}{\partial \mu_{L-1}}(\mu_1^*, \dots, \mu_{L-1}^*). \end{aligned}$$

If $(\mu_1^*, \dots, \mu_L^*)$ also satisfies the first-order condition for μ_{L-1} , then $\frac{\partial J_{L-1}}{\partial \mu_{L-1}}(\mu_1^*, \dots, \mu_{L-1}^*) = 0$. Continuing this telescoping argument, we conclude if $(\mu_1^*, \dots, \mu_L^*)$ satisfies the first-order condition for all μ_i , $1 \leq i \leq L$, then $\frac{\partial J_i}{\partial \mu_i}(\mu_1^*, \dots, \mu_i^*) = 0$ for every $1 \leq i \leq L$.

Given the form of J_1 , it is clear that $\frac{\partial J_1}{\partial \mu_1}(\mu_1^*) = 0$ implies $\mu_1^* = \mu_1^\bullet$. Also,

$$\frac{\partial J_i}{\partial \mu_i}(\mu_1^*, \dots, \mu_i^*) = - \int_{-\infty}^{c_1} \dots \int_{-\infty}^{c_{i-1}(x_1, \dots, x_{i-2})} \prod_{j=1}^{i-1} \phi(x_j; \mu_j^\bullet, \sigma^2) \left[\frac{(\mu_i^\bullet - \mu_i^* + \sum \gamma_{i,j} \cdot (x_j - \mu_j^*))}{\sigma^2} \right] dx_{i-1} \dots dx_1.$$

Using the fact that $\frac{\partial J_i}{\partial \mu_i}(\mu_1^*, \dots, \mu_i^*) = 0$, we multiply the integrand by the constant

$$-\sigma^2 \cdot \left(\int_{-\infty}^{c_1} \dots \int_{-\infty}^{c_{i-1}(x_1, \dots, x_{i-2})} \prod_{j=1}^{i-1} \phi(x_j; \mu_j^\bullet, \sigma^2) dx_{i-1} \dots dx_1 \right)^{-1}$$

and get

$$\mathbb{E}_{\Psi^\bullet} \left[\mu_i^\bullet - \mu_i^* + \sum_{j=1}^{i-1} \gamma_{i,j} \cdot (X_j - \mu_j^*) \mid (X_k)_{k=1}^{i-1} \in R_{i-1} \right] = 0.$$

Rearranging, we have $\mu_i^* = \mu_i^\bullet - \sum_{j=1}^{i-1} \gamma_{i,j} \cdot (\mu_j^* - \mathbb{E}_{\Psi^\bullet}[X_j \mid (X_k)_{k=1}^{i-1} \in R_{i-1}])$ as desired. This means the only $(\mu_1^*, \dots, \mu_L^*)$ satisfying the first-order condition for minimizing KL divergence is the one iteratively given in this proposition. \square

OA 1.9 Proof of Proposition A.7

Proof. This clearly holds for $i = 1$. By induction assume this holds for all $i \leq K$ for some $K \leq L - 1$. I show that this also holds for $i = K + 1$.

From Proposition A.6,

$$\mu_i^* = \mu_i^\bullet - \sum_{j=1}^{i-1} \gamma_{i,j} \cdot (\mu_j^* - \mathbb{E}_{\Psi^\bullet}[X_j | (X_k)_{k=1}^{i-1} \in R_{i-1}]).$$

The continuation region R_{i-1} is the rectangle $(-\infty, c_1) \times \dots \times (-\infty, c_{i-1}) \in \mathbb{R}^{i-1}$. As (X_1, \dots, X_{i-1}) are objectively independent, the events $\{X_k \leq c_k\}$ for $k \neq j$ are independent of X_j , so the expression simplifies to

$$\mu_i^* = \mu_i^\bullet - \sum_{j=1}^{i-1} \gamma_{i,j} \cdot (\mu_j^* - \mathbb{E}_{\Psi^\bullet}[X_j | X_j \leq c_j]).$$

Expanding each μ_j^* for $1 \leq j \leq i-1$ using the inductive hypothesis,

$$\begin{aligned} \mu_{K+1}^* &= \mu_{K+1}^\bullet - \sum_{j=1}^K \gamma_{K+1,j} \cdot (\mu_j^\bullet - \mathbb{E}_{\Psi^\bullet}[X_j | X_j \leq c_j]) \\ &\quad + \sum_{j=1}^K -\gamma_{K+1,j} \cdot \left(\sum_{k=1}^{j-1} \left(\sum_{p \in P[j \rightarrow k]} W(p) \right) \cdot (\mu_k^\bullet - \mathbb{E}_{\Psi^\bullet}[X_k | X_k \leq c_k]) \right) \\ &= \mu_{K+1}^\bullet + \sum_{j=1}^K \left[(-\gamma_{K+1,j}) + \sum_{k=j+1}^K -\gamma_{K+1,k} \cdot \left(\sum_{p \in P[k \rightarrow j]} W(p) \right) \right] \cdot (\mu_j^\bullet - \mathbb{E}_{\Psi^\bullet}[X_j | X_j \leq c_j]). \end{aligned}$$

Paths in $P[K+1 \rightarrow j]$ come in two types. The first type is the direct path consisting of just one edge $(K+1, j)$, with weight $-\gamma_{K+1,j}$. The second type consists of the indirect paths $p = ((K+1, k), p')$ where $p' \in P[k \rightarrow j]$. We have $W(p) = -\gamma_{K+1,k} \cdot W(p')$. We therefore see that the expression $\sum_{j=1}^K [(-\gamma_{K+1,j}) + \sum_{k=j+1}^K -\gamma_{K+1,k} \cdot (\sum_{p \in P[k \rightarrow j]} W(p))]$ in fact gives the sum of weights for all paths in $P[K+1 \rightarrow j]$. So, we have shown that the claim holds also for $i = K+1$. By induction it holds for all $1 \leq i \leq L$. \square

OA 1.10 Proof of Corollary A.2

Proof. First suppose $\delta > \alpha$. By Proposition A.7, since $\mu_j^\bullet - \mathbb{E}[X_j | X_j \leq c_j] > 0$ for any $c_j \in \mathbb{R}$, I only need to show that $\sum_{p \in P[i \rightarrow j]} W(p) < 0$ for every $i > j$ pair. Due to the stationarity of γ under the $\gamma_{i,j} = \alpha \cdot \delta^{i-j-1}$ functional form, it suffices to prove $\sum_{p \in P[i \rightarrow 1]} W(p) < 0$ for every $2 \leq i \leq L$.

When $i = 2$, $P[2 \rightarrow 1]$ consists of a single path with weight $-\alpha < 0$. By induction suppose $\sum_{p \in P[i \rightarrow 1]} W(p) < 0$ for all $i \leq S$ for $2 \leq S \leq L-1$. We can exhaustively enumerate $p \in P[S+1 \rightarrow 1]$ by relating each path in $P[S \rightarrow 1]$ to a pair of paths in $P[S+1 \rightarrow 1]$. Relate $p = ((S, i_1), \dots, (i_{M-1}, 1)) \in P[S \rightarrow 1]$ to the pair $p' = ((S+1, i_1), \dots, (i_{M-1}, 1))$ and $p'' = ((S+1, S), (S, i_1), \dots, (i_{M-1}, 1))$. That is, p' modifies the first edge in p from (S, i_1) to

$(S + 1, i_1)$, while p'' simply concatenates the extra edge $(S + 1, S)$ in front of p . We have $W(p') = \delta \cdot W(p)$, because the weight of (S, i_1) is $-\alpha\delta^{S-i_1-1}$ while the weight of $(S + 1, i_1)$ is $-\alpha\delta^{S-i_1}$, and the two paths are otherwise identical. We have $W(p'') = -\alpha \cdot W(p)$, since the newly concatenated edge has weight $-\alpha$. This argument shows $\sum_{p \in \mathcal{P}[S+1 \rightarrow 1]} W(p) = (\delta - \alpha) \cdot \sum_{p \in \mathcal{P}[S \rightarrow 1]} W(p)$. Since $\delta - \alpha > 0$ and $\sum_{p \in \mathcal{P}[S \rightarrow 1]} W(p) < 0$ by the inductive hypothesis, we also have $\sum_{p \in \mathcal{P}[S+1 \rightarrow 1]} W(p) < 0$. By induction, we have shown that $\sum_{p \in \mathcal{P}[i \rightarrow 1]} W(p) < 0$ for every $2 \leq i \leq L$.

Next, suppose $\delta < \alpha$. By Proposition A.7,

$$\mu_3^* = \mu_3^\bullet + (-\alpha\delta + \alpha^2) \cdot (\mu_1^\bullet - \mathbb{E}[X_1|X_1 \leq c_1]) + (-\alpha) (\mu_2^\bullet - \mathbb{E}[X_2|X_2 \leq c_2]).$$

The coefficient in front of $\mu_1^\bullet - \mathbb{E}[X_1|X_1 \leq c_1]$ comes from the fact that there are two paths from 3 to 1, with weights $-\gamma_{3,1} = -\alpha\delta$ and $(-\gamma_{3,2}) \cdot (-\gamma_{2,1}) = (-\alpha) \cdot (-\alpha) = \alpha^2$. We have $(-\alpha\delta + \alpha^2) = \alpha(\alpha - \delta) > 0$ since $\alpha > 0$ and $\delta < \alpha$. So, fixing c_2 , as $c_1 \rightarrow -\infty$ we get $\mu_1^\bullet - \mathbb{E}[X_1|X_1 \leq c_1] \rightarrow \infty$ and therefore $\mu_3^* \rightarrow \infty$. \square

OA 1.11 Proof of Proposition A.9

Proof. Write $\phi(x; a, b^2)$ for the Gaussian density with mean a , variance b^2 , evaluated at x . Without loss, suppose $h_{2,n} \neq \emptyset$ for all $1 \leq n \leq N_1$, and $h_{2,n} = \emptyset$ for all $n > N_1$. I show that the posterior density over (μ_1, μ_2) after the dataset $(h_n)_{n=1}^N$ only depends on $N_1, \frac{1}{N} \sum_{n=1}^N h_{1,n}$, and $\frac{1}{N_1} \sum_{n=1}^{N_1} (h_{2,n} + \gamma h_{1,n})$. Indeed,

$$\begin{aligned} g(\mu_1, \mu_2 | (h_n)_{n=1}^N) &\propto g(\mu_1, \mu_2) \cdot \left[\prod_{n=1}^{N_1} \phi(h_{1,n}; \mu_1, \sigma^2) \cdot \phi(h_{2,n}; \mu_2 - \gamma(h_{1,n} - \mu_1), \sigma^2) \right] \cdot \left[\prod_{n=N_1+1}^N \phi(h_{1,n}; \mu_1, \sigma^2) \right] \\ &= g(\mu_1, \mu_2) \cdot \left[\prod_{n=1}^N \phi(h_{1,n}; \mu_1, \sigma^2) \right] \times \left[\prod_{n=1}^{N_1} \phi(h_{2,n}; \mu_2 - \gamma(h_{1,n} - \mu_1), \sigma^2) \right] \\ &= \left[\prod_{n=1}^N \phi(h_{1,n}; \mu_1, \sigma^2) \right] \times \left[\prod_{n=1}^{N_1} \phi(h_{2,n} + \gamma h_{1,n}; \mu_2 + \gamma \mu_1, \sigma^2) \right]. \end{aligned}$$

It is well-known that under the Gaussian likelihood, $(h_{1,n})_{n=1}^N \mapsto \prod_{n=1}^N \phi(h_{1,n}; \mu_1, \sigma^2)$ is a function of $\frac{1}{N} \sum_{n=1}^N h_{1,n}$, and for the same reason $(h_{2,n} + \gamma h_{1,n})_{n=1}^{N_1} \mapsto \prod_{n=1}^{N_1} \phi(h_{2,n} + \gamma h_{1,n}; \mu_2 + \gamma \mu_1, \sigma^2)$ is a function of $\frac{1}{N_1} \sum_{n=1}^{N_1} (h_{2,n} + \gamma h_{1,n})$.

Since the posterior belief $g(\cdot | (h_n)_{n=1}^N)$ only depends on N_1 and the two statistics $\Lambda_1^{(N)}((h_n)_{n=1}^N)$, $\Lambda_2^{(N)}((h_n)_{n=1}^N) \in \mathbb{R}$, the optimal cutoff rule may be expressed as a function of these two statistics, N_1 , and c of the predecessors. \square

OA 1.12 Proof of Proposition A.10

Proof. To see the first two equations, let $c \in \mathbb{R}$, $\mu_1, \mu_2 \in \mathbb{R}$, and write $\Psi = \Psi(\mu_1, \mu_2; \gamma)$. We have

$$\begin{aligned}
& \mathbb{E}_{h \sim \mathcal{H}(\Psi; c)}[h_2 + \gamma h_1 \mid h_2 \neq \emptyset] \\
&= \mathbb{E}_{h \sim \mathcal{H}(\Psi; c)}[h_2 \mid h_2 \neq \emptyset] + \gamma \mathbb{E}_{h \sim \mathcal{H}(\Psi; c)}[h_1 \mid h_2 \neq \emptyset] \\
&= \mathbb{E}_{\Psi}[X_2 \mid X_1 \leq c] + \gamma \mathbb{E}_{\Psi}[X_1 \mid X_1 \leq c] \\
&= \mathbb{E}_{\Psi}[\mu_2 - \gamma(X_1 - \mu_1) \mid X_1 \leq c] + \gamma \mathbb{E}_{\Psi}[X_1 \mid X_1 \leq c] \\
&= \mathbb{E}_{\Psi}[\mu_2 + \gamma \mu_1 \mid X_1 \leq c] \\
&= \mu_2 + \gamma \mu_1.
\end{aligned}$$

Since this holds for any c , so we must get that on the mixed history distribution, $\Lambda_2(\mathcal{H}(\Psi(\mu_1, \mu_2, \gamma); c_1, \dots, c_L))$ $\mu_2 + \gamma \mu_1$ as well. It is easy to see that we must have $\Lambda_1(\mathcal{H}(\Psi(\mu_1, \mu_2, \gamma); c_1, \dots, c_L)) = \mu_1$.

To obtain the final equation, first note that we can re-write the second statistic under the true distribution of histories $\mathcal{H}^\bullet(c_1, \dots, c_L)$ as a weighted average,

$$\mathbb{E}_{h \sim \mathcal{H}^\bullet(c_1, \dots, c_L)}[h_2 + \gamma h_1 \mid h_2 \neq \emptyset] = \sum_{\ell=1}^L w_\ell \cdot \mathbb{E}_{h \sim \mathcal{H}^\bullet(c_\ell)}[h_2 + \gamma h_1 \mid h_2 \neq \emptyset]$$

This is because the event of $h_2 \neq \emptyset$ happens only when h_1 falls below the censoring threshold, so the posterior probability of h being generated from the sub-distribution $\mathcal{H}^\bullet(c_\ell)$ given that $h_2 \neq \emptyset$ depends on the relative likelihoods of X_1 falling under the L different censoring thresholds.

For each ℓ ,

$$\mathbb{E}_{h \sim \mathcal{H}^\bullet(c_\ell)}[h_2 + \gamma h_1 \mid h_1 \leq c] = \mu_2^\bullet + \gamma \mathbb{E}[X_1 \mid X_1 \leq c_\ell]$$

where the conditional expectation of $h_2 \mid h_1 \leq c_\ell$ is simply μ_2^\bullet by independence of X_2 and X_1 under Ψ^\bullet . Putting this into the weighted average expression,

$$\mathbb{E}_{h \sim \mathcal{H}^\bullet(c_1, \dots, c_L)}[h_2 + \gamma h_1 \mid h_2 \neq \emptyset] = \mu_2^\bullet + \gamma \sum_{\ell=1}^L w_\ell \cdot \mathbb{E}[X_1 \mid X_1 \leq c_\ell].$$

In order to match the statistics, $s_1 = \mu_1^\bullet$ and $s_2 = \mu_2^\bullet + \gamma \sum_{\ell=1}^L w_\ell \cdot \mathbb{E}[X_1 \mid X_1 \leq c_\ell]$ produced by $\Lambda(\mathcal{H}^\bullet(c_1, \dots, c_L))$, we must therefore have $\mu_1 = \mu_1^\bullet$, and

$$\mu_2^\bullet + \gamma \sum_{\ell=1}^L w_\ell \cdot \mathbb{E}[X_1 \mid X_1 \leq c_\ell] = \mu_2 + \gamma \mu_1^\bullet,$$

which rearranges to

$$\mu_2 = \mu_2^\bullet - \gamma \sum_{\ell=1}^L w_\ell \cdot (\mu_1^\bullet - \mathbb{E}[X_1 \mid X_1 \leq c_\ell]) = \mu_2^*(c_1, \dots, c_L).$$

□

OA 2 Proof of Theorem 1

In this section I prove the almost-sure convergence of beliefs and behavior when biased agents act one at a time and entertain uncertainty over both μ_1 and μ_2 .

For $\underline{\mu}_1 < \bar{\mu}_1, \underline{\mu}_2 < \bar{\mu}_2$, let $\diamond([\underline{\mu}_1, \bar{\mu}_1], [\underline{\mu}_2, \bar{\mu}_2])$ refer to the parallelogram in \mathbb{R}^2 with the vertices:

- $(\underline{\mu}_1, \bar{\mu}_2 + \frac{\gamma}{2}(\bar{\mu}_1 - \underline{\mu}_1))$
- $(\underline{\mu}_1, \underline{\mu}_2 + \frac{\gamma}{2}(\bar{\mu}_1 - \underline{\mu}_1))$
- $(\bar{\mu}_1, \bar{\mu}_2 - \frac{\gamma}{2}(\bar{\mu}_1 - \underline{\mu}_1))$
- $(\bar{\mu}_1, \underline{\mu}_2 - \frac{\gamma}{2}(\bar{\mu}_1 - \underline{\mu}_1))$

In other words, $\diamond([\underline{\mu}_1, \bar{\mu}_1], [\underline{\mu}_2, \bar{\mu}_2])$ is the parallelogram constructed by starting with the rectangle $[\underline{\mu}_1, \bar{\mu}_1] \times [\underline{\mu}_2, \bar{\mu}_2]$, then replacing the top and bottom edges with lines with slope $-\gamma$ (and adjusting the left and right edges accordingly to connect with the new top and bottom edges.)

Consider a sequence of short-lived agents playing the stage game in rounds $t = 1, 2, 3, \dots$. They are uncertain about both μ_1 and μ_2 , with a prior density $g(\mu_1, \mu_2)$ supported on feasible fundamentals $\mathcal{M} = \diamond([\underline{\mu}_1, \bar{\mu}_1], [\underline{\mu}_2, \bar{\mu}_2])$ as in Remark 1(b). I abbreviate this support as \diamond when no confusion arises. Each agent t choose the the optimal cutoff \tilde{C}_t maximizing expected payoff based on posterior belief formed from all past histories. I show the almost sure convergence of stochastic processes (\tilde{C}_t) and (\tilde{G}_t) to the unique steady state under the hypotheses of Theorem 1.

OA 2.1 Preliminary Results

First, I consider how the predicted second-period payoff after $X_1 = x_1$ depends on the parameters of the subjective model $\Psi(\mu_1, \mu_2; \gamma)$.

Lemma OA.2. For every $\mu_1, \mu_2, x_1 \in \mathbb{R}$, the conditional distribution $X_2|X_1 = x_1$ is the same under $\Psi(\mu_1^\bullet, \mu_2 + \gamma(\mu_1 - \mu_1^\bullet); \gamma)$ and $\Psi(\mu_1, \mu_2; \gamma)$. So in particular, $C(\mu_1, \mu_2; \gamma) = C(\mu_1^\bullet, \mu_2 + \gamma(\mu_1 - \mu_1^\bullet); \gamma)$.

Proof. Under the subjective model $\Psi(\mu_1^\bullet, \mu_2 + \gamma(\mu_1 - \mu_1^\bullet); \gamma)$, the conditional distribution of X_2 given $X_1 = x_1$ is $\mathcal{N}(\mu_2 + \gamma(\mu_1 - \mu_1^\bullet) - \gamma(x_1 - \mu_1^\bullet), \sigma^2)$, which simplifies to $\mathcal{N}(\mu_2 - \gamma(x_1 - \mu_1); \sigma^2)$. It is easy to see that this is also the expression for the same conditional distribution under $\Psi(\mu_1, \mu_2; \gamma)$.

Suppose $C(\mu_1, \mu_2; \gamma) = c$. This implies the indifference condition,

$$u_1(c) = \mathbb{E}_{\Psi(\mu_1, \mu_2; \gamma)}[u_2(c, X_2) \mid X_1 = c].$$

But by the equivalence of conditional distribution given above,

$$u_1(c) = \mathbb{E}_{\Psi(\mu_1^\bullet, \mu_2 + \gamma(\mu_1 - \mu_1^\bullet); \gamma)}[u_2(c, X_2) \mid X_1 = c].$$

This means c is also the indifference threshold for the model $\Psi(\mu_1^\bullet, \mu_2 + \gamma(\mu_1 - \mu_1^\bullet); \gamma)$. \square

As a corollary, this lemma shows the restriction to cutoff strategies is without loss, and that \tilde{C}_t is well defined. That is, for any belief given by a density on \mathcal{M} , there exists a cutoff strategy that is weakly optimal among the class of all stopping strategies, and further this cutoff strategy is strictly optimal among the class of cutoff strategies. This is because for any $x_1 \in \mathbb{R}$ and any density \tilde{g} on \mathcal{M} ,

$$\int_{\mathcal{M}} \mathbb{E}_{\Psi(\mu_1, \mu_2; \gamma)}[u_2(x_1, X_2) \mid X_1 = x_1] \cdot \tilde{g}(\mu_1, \mu_2) d(\mu_1, \mu_2) = \int_{\underline{\mu}_2^\circ}^{\bar{\mu}_2^\circ} \mathbb{E}_{\Psi(\mu_1^\bullet, \mu_2; \gamma)}[u_2(x_1, X_2) \mid X_1 = x_1] \cdot \tilde{g}^V(\mu_2) d\mu_2$$

where $\bar{\mu}_2^\circ := \max\{\mu_2 : (\mu_1^\bullet, \mu_2) \in \diamond\}$ and $\underline{\mu}_2^\circ := \min\{\mu_2 : (\mu_1^\bullet, \mu_2) \in \diamond\}$, and $\tilde{g}^V(\mu_2)$ is the integral of $\tilde{g}(\mu_1, \mu_2)$ over the line in \diamond with slope $-\gamma$ that passes through (μ_1^\bullet, μ_2) . This equality holds because by Lemma OA.2, all fundamentals on that line imply the same continuation payoff after $X_1 = x_1$ as the fundamentals (μ_1^\bullet, μ_2) . The proof of Lemma A.5 shows that

$$x_1 \mapsto u_1(x_1) - \int_{\underline{\mu}_2^\circ}^{\bar{\mu}_2^\circ} \mathbb{E}_{\Psi(\mu_1^\bullet, \mu_2; \gamma)}[u_2(x_1, X_2) \mid X_1 = x_1] \tilde{g}^V(\mu_2) d\mu_2$$

is a strictly increasing, continuous function that crosses 0.

Now, the key step is to separate the two-dimensional inference problem into a pair of one-dimensional problems.

OA 2.2 Learning μ_1^\bullet

I define the stochastic process of data log-likelihood (for a given fundamental). For each $\mu_1, \mu_2 \in \text{supp}(g)$, let $\ell_t(\mu_1, \mu_2)(\omega)$ be the log likelihood that the fundamentals are (μ_1, μ_2) and histories $(\tilde{H}_s)_{s \leq t}(\omega)$ are generated by the end of round t . It is given by

$$\ell_t(\mu_1, \mu_2)(\omega) := \ln(g(\mu_1, \mu_2)) + \sum_{s=1}^t \ln(\text{lik}(\tilde{H}_s(\omega); \mu_1, \mu_2))$$

where $\text{lik}(x_1, \emptyset; \mu_1, \mu_2) := \phi(x_1; \mu_1, \sigma^2)$ and $\text{lik}(x_1, x_2; \mu_1, \mu_2) := \phi(x_1; \mu_1, \sigma^2) \cdot \phi(x_2; \mu_2 - \gamma(x_1 - \mu_1); \sigma^2)$. By simple algebra, we may expand

$$\begin{aligned} \ell_t(\mu_1, \mu_2)(\omega) &= \ln(g(\mu_1, \mu_2)) + \sum_{s=1}^t \left[\ln((2\pi\sigma^2)^{-1/2}) \right] - \frac{(X_{1,s}(\omega) - \mu_1)^2}{2\sigma^2} \\ &\quad + \sum_{s=1}^t \mathbf{1}\{X_{1,s}(\omega) \leq \tilde{C}_s(\omega)\} \cdot \left(\ln((2\pi\sigma^2)^{-1/2}) - \frac{(X_{2,s}(\omega) - \mu_2 + \gamma(X_{1,s}(\omega) - \mu_1))^2}{2\sigma^2} \right) \end{aligned}$$

I first establish that, without knowing anything about the process (C_t) , we can conclude agents learn μ_1^\bullet arbitrarily well.

Lemma OA.3. *For every $\epsilon > 0$, almost surely $\lim_{t \rightarrow \infty} \tilde{G}_t(\diamond \cap ([\mu_1^\bullet - \epsilon, \mu_1^\bullet + \epsilon] \times \mathbb{R})) = 1$.*

Proof. I first calculate the directional derivative

$$\nabla_v \frac{1}{t} \ell_t(\mu_1, \mu_2),$$

where $v = \begin{pmatrix} 1/\sqrt{1+\gamma^2} \\ -\gamma/\sqrt{1+\gamma^2} \end{pmatrix}$ is the unit vector with slope $-\gamma$. We have

$$\frac{\partial(\ell_t/t)}{\partial\mu_1}(\mu_1, \mu_2) = \frac{1}{t} \frac{D_1 g(\mu_1, \mu_2)}{g(\mu_1, \mu_2)} + \frac{1}{\sigma^2} \left(\frac{1}{t} \sum_{s=1}^t X_{1,s} - \mu_1 \right) + \frac{\gamma}{\sigma^2} \frac{1}{t} \sum_{s=1}^t \mathbf{1}\{X_{1,s} \leq \tilde{C}_s\} \cdot (X_{2,s} - \mu_2 + \gamma(X_{1,s} - \mu_1))$$

$$\frac{\partial(\ell_t/t)}{\partial\mu_2}(\mu_1, \mu_2) = \frac{1}{t} \frac{D_2 g(\mu_1, \mu_2)}{g(\mu_1, \mu_2)} + \frac{1}{\sigma^2} \frac{1}{t} \sum_{s=1}^t \mathbf{1}\{X_{1,s} \leq \tilde{C}_s\} \cdot (X_{2,s} - \mu_2 + \gamma(X_{1,s} - \mu_1)),$$

where $D_1 g$ and $D_2 g$ are the two partial derivatives of g . At every ω and every (μ_1, μ_2) , note the last summand in $\frac{\partial(\ell_t/t)}{\partial\mu_1}$ is γ times the last summand in $\frac{\partial(\ell_t/t)}{\partial\mu_2}$. Therefore,

$$\nabla_v \frac{1}{t} \ell_t(\mu_1, \mu_2) = \frac{1}{\sigma^2 \sqrt{1+\gamma^2}} \left(\frac{1}{t} \sum_{s=1}^t X_{1,s} - \mu_1 \right) + \frac{1}{t \sqrt{1+\gamma^2}} \frac{1}{t} \frac{D_1 g(\mu_1, \mu_2)}{g(\mu_1, \mu_2)} - \frac{\gamma}{t \sqrt{1+\gamma^2}} \frac{D_2 g(\mu_1, \mu_2)}{g(\mu_1, \mu_2)}.$$

Since g, D_1g, D_2g are continuous on the compact set \diamond , there exists some $0 < B < \infty$ so that $|\frac{D_1g(\mu_1, \mu_2)}{g(\mu_1, \mu_2)}| < B$ and $|\frac{D_2g(\mu_1, \mu_2)}{g(\mu_1, \mu_2)}| < B$ for all $(\mu_1, \mu_2) \in \diamond$. This means for every ω ,

$$\inf_{(\mu_1, \mu_2) \in \diamond_\ell} \left(\nabla_v \frac{1}{t} \ell_t(\mu_1, \mu_2) \right) \geq \frac{1}{\sigma^2 \sqrt{1 + \gamma^2}} \left(\frac{1}{t} \sum_{s=1}^t X_{1,s} - (\mu_1^\bullet - \epsilon) \right) + \frac{1}{t} \frac{2(1 + \gamma)}{\sqrt{1 + \gamma^2}} B,$$

where $\diamond_L := \diamond \cap ([\underline{\mu}_1, \mu_1^\bullet - \epsilon] \times \mathbb{R})$ is the sub-parallelogram to the left of $\mu_1^\bullet - \epsilon$. By law of large numbers applied to the i.i.d. sequence $(X_{1,s})$, almost surely $\frac{1}{t} \sum_{s=1}^t X_{1,s} \rightarrow \mu_1^\bullet$, therefore almost surely

$$\liminf_{t \rightarrow \infty} \inf_{(\mu_1, \mu_2) \in \diamond_L} \left(\nabla_v \frac{1}{t} \ell_t(\mu_1, \mu_2) \right) \geq \frac{\epsilon}{\sigma^2 \sqrt{1 + \gamma^2}}.$$

We may divide \diamond_L further divide into two halves:

$$\diamond_{L,1} := \diamond \cap ([\underline{\mu}_1, \underline{\mu}_1 + d/2] \times \mathbb{R})$$

$$\diamond_{L,2} := \diamond \cap ([\underline{\mu}_1 + d/2, \mu_1^\bullet - \epsilon] \times \mathbb{R})$$

where $d := \mu_1^\bullet - \epsilon - \underline{\mu}_1$. I will show that $\lim_{t \rightarrow \infty} \tilde{G}_t(\diamond_{L,1}) = 0$ almost surely. The idea is we can map every point in $\diamond_{L,1}$ to another point in $\diamond_{L,2}$ in the direction of v . For every point, its image under the map will have much higher posterior probability, since we have a uniform, strictly positive lowerbound on the directional derivative of log-likelihood ℓ_t in the direction of v .

$$\begin{aligned} \tilde{G}_t(\diamond_{L,1}) &= \int_{\diamond_{L,1}} \tilde{g}_t(\mu_1, \mu_2) d\mu \\ &= \int_{\diamond_{L,2}} \tilde{g}_t(\mu_1, \mu_2) \cdot \frac{\tilde{g}_t(\mu_1 - d, \mu_2 - \gamma d)}{\tilde{g}_t(\mu_1, \mu_2)} d\mu \\ &= \int_{\diamond_{L,2}} \tilde{g}_t(\mu_1, \mu_2) \exp(\ell_t(\mu_1 - d, \mu_2 - \gamma d) - \ell_t(\mu_1, \mu_2)) d\mu \\ &= \int_{\diamond_{L,2}} \tilde{g}_t(\mu_1, \mu_2) \exp\left(-\int_0^d \nabla_v \ell_t(\mu_1 - d + z, \mu_2 - \gamma d + \gamma z) dz\right) d\mu \end{aligned}$$

Almost surely,

$$\liminf_{t \rightarrow \infty} \inf_{(\mu_1, \mu_2) \in \diamond_{L,2}, z \in [0, d]} (\nabla_v \ell_t(\mu_1 - d + z, \mu_2 - \gamma d + \gamma z)) \geq \frac{t\epsilon}{\sigma^2 \sqrt{1 + \gamma^2}},$$

so almost surely

$$\limsup_{t \rightarrow \infty} \tilde{G}_t(\diamond_{L,1}) \leq \limsup_{t \rightarrow \infty} \int_{\diamond_{L,2}} \tilde{g}_t(\mu_1, \mu_2) \exp\left(-\frac{dt\epsilon}{\sigma^2 \sqrt{1 + \gamma^2}}\right) d\mu.$$

But for every ω and t , the RHS is bounded by $\exp(-\frac{dt\epsilon}{\sigma^2\sqrt{1+\gamma^2}})$, which tends to 0 as $t \rightarrow \infty$. So in fact $\tilde{G}_t(\diamond_{L,1}) \rightarrow 0$ almost surely.

Now by dividing $\diamond_{L,2}$ into two equal halves and iterating this argument, we eventually show $\lim_{t \rightarrow \infty} \tilde{G}_t(\diamond \cap ([\mu_1^\bullet - \epsilon, \infty) \times \mathbb{R})) = 1$. A symmetric argument also shows $\lim_{t \rightarrow \infty} \tilde{G}_t(\diamond \cap ((-\infty, \mu_1^\bullet + \epsilon] \times \mathbb{R})) = 1$. \square

OA 2.3 Decomposing Partial Derivative of Log-Likelihood With Respect to μ_2

I record a decomposition of $\frac{\partial \ell}{\partial \mu_2}(\mu_1, \mu_2)$, the partial derivative of the log-likelihood process with respect to its second argument.

Define two stochastic processes:

$$\varphi_s(\mu_1, \mu_2) := \sigma^{-2} \cdot (X_{2,s} - \mu_2 + \gamma(X_{1,s} - \mu_1)) \cdot \mathbf{1}\{X_{1,s} \leq \tilde{C}_s\}$$

$$\bar{\varphi}_s(\mu_1, \mu_2) := \sigma^{-2} \mathbb{P}[X_1 \leq \tilde{C}_s] \cdot (\mu_2^\bullet - \mu_2 - \gamma(\mu_1 - \mathbb{E}[X_1 | X_1 \leq \tilde{C}_s])),$$

with a slight abuse of notation, $\mathbb{P}[X_1 \leq x]$ means the probability that each first-period draw falls below x , and $\mathbb{E}[X_1 | X_1 \leq x]$ the conditional expectation of the first draw given that it falls below x . Note that $\bar{\varphi}_s(\mu_1, \mu_2)$ is measurable with respect to \mathcal{F}_{s-1} , since (\tilde{C}_t) is a predictable process. Write $\xi_s(\mu_1, \mu_2) := \varphi_s(\mu_1, \mu_2) - \bar{\varphi}_s(\mu_1, \mu_2)$ and $y_t(\mu_1, \mu_2) := \sum_{s=1}^t \xi_s(\mu_1, \mu_2)$. Write $z_t(\mu_1, \mu_2) := \sum_{s=1}^t \bar{\varphi}_s(\mu_1, \mu_2)$.

Lemma OA.4. $\frac{\partial \ell_t}{\partial \mu_2}(\mu_1, \mu_2) = \frac{D_2 g(\mu_1, \mu_2)}{g(\mu_1, \mu_2)} + y_t(\mu_1, \mu_2) + z_t(\mu_1, \mu_2)$

Proof. This comes from expanding $\ell_t(\mu_1, \mu_2)$ and taking its derivative as in the proof of Lemma OA.3. \square

Now I derive two results about the $\xi_t(\mu_1, \mu_2)$ processes for different pairs (μ_1, μ_2) .

Lemma OA.5. *There exists $\kappa_\xi < \infty$ so that for every $(\mu_1, \mu_2) \in \diamond$ and for every $t \geq 1$, $\omega \in \Omega$, $\mathbb{E}[\xi_t^2(\mu_1, \mu_2) | \mathcal{F}_{t-1}](\omega) \leq \kappa_\xi$.*

Proof. Note that $\bar{\varphi}_t(\mu_1, \mu_2)$ is measurable with respect to \mathcal{F}_{t-1} . Also, $\varphi_t(\mu_1, \mu_2) | \mathcal{F}_{t-1} = \varphi_t(\mu_2) | \tilde{C}_t$, because by independence of X_t from $(X_s)_{s=1}^{t-1}$, the only information that \mathcal{F}_{t-1} contains about $\varphi_t(\mu_1, \mu_2)$ is in determining the cutoff threshold \tilde{C}_t .

At a sample path ω so that $\tilde{C}_t(\omega) = c \in \mathbb{R}$,

$$\begin{aligned} \mathbb{E}[\varphi_s(\mu_1, \mu_2) | \mathcal{F}_{t-1}](\omega) &= \mathbb{E}[\sigma^{-2} \cdot (X_2 - \mu_2 + \gamma(X_1 - \mu_1)) \mathbf{1}\{X_1 \leq c\}] \\ &= \sigma^{-2} (\mathbb{E}[(X_2 - \mu_2) \cdot \mathbf{1}\{X_1 \leq c\}] + \mathbb{E}[\gamma(X_1 - \mu_1) \cdot \mathbf{1}\{X_1 \leq c\}]) \\ &= \sigma^{-2} \mathbb{P}[X_1 \leq c] \cdot (\mu_2^\bullet - \mu_2 - \gamma(\mu_1 - \mathbb{E}[X_1 | X_1 \leq c])), \end{aligned}$$

where we used the fact that $X_{1,t}$ and $X_{2,t}$ are independent. This shows that $\mathbb{E}[\varphi_s(\mu_1, \mu_2)|\mathcal{F}_{t-1}](\omega) = \bar{\varphi}_s(\mu_1, \mu_2)(\omega)$. Since this holds regardless of c , we get that $\mathbb{E}[\varphi_s(\mu_1, \mu_2)|\mathcal{F}_{t-1}] = \bar{\varphi}_t(\mu_1, \mu_2)$ for all ω , that is to say

$$\begin{aligned} \mathbb{E}[\xi_t^2(\mu_1, \mu_2)|\mathcal{F}_{t-1}] &= \text{Var}[\varphi_t(\mu_1, \mu_2)|\mathcal{F}_{t-1}] \\ &\leq \mathbb{E}[\varphi_t^2(\mu_1, \mu_2)|\mathcal{F}_{t-1}] \\ &= \sigma^{-4} \cdot \mathbb{E}[(X_{2,s} - \mu_2 + \gamma(X_{1,s} - \mu_1))^2 \cdot \mathbf{1}\{X_{1,s} \leq \tilde{C}_s\}] \\ &\leq \sigma^{-4} \cdot \mathbb{E}[(X_2 - \mu_2 + \gamma(X_1 - \mu_1))^2] \end{aligned}$$

The RHS of the final line is independent of ω and t , while $(\mu_1, \mu_2) \mapsto \mathbb{E}[(X_2 - \mu_2 + \gamma(X_1 - \mu_1))^2]$ is a continuous function on \diamond . Therefore it is bounded uniformly by some $\kappa_\xi < \infty$, which also provides a bound for $\mathbb{E}[\xi_t^2(\mu_1, \mu_2)|\mathcal{F}_{t-1}](\omega)$ for every t, ω and $(\mu_1, \mu_2) \in \diamond$. \square

OA 2.4 **Heidhues, Koszegi, and Strack (2018)**'s Law of Large Numbers

I use a statistical result from **Heidhues, Koszegi, and Strack (2018)** to show that the y_t/t term in the decomposition of $\frac{1}{t} \frac{\partial \ell_t}{\partial \mu_2}$ almost surely converges to 0 in the long run, and furthermore this convergence is uniform on \diamond . This lets me focus on terms of the form $\bar{\varphi}_s(\mu_1, \mu_2)$, which can be interpreted as the *expected* contribution to the log likelihood derivative from round s data. This lends tractability to the problem as $\bar{\varphi}_s(\mu_1, \mu_2)$ only depends on \tilde{C}_s , but not on $X_{1,s}$ or $X_{2,s}$.

Lemma OA.6. *For every $(\mu_1, \mu_2) \in \diamond$, $\lim_{t \rightarrow \infty} |\frac{y_t(\mu_1, \mu_2)}{t}| = 0$ almost surely.*

Proof. **Heidhues, Koszegi, and Strack (2018)**'s Proposition 10 shows that if (y_t) is a martingale such that there exists some constant $v \geq 0$ satisfying $[y]_t \leq vt$ almost surely, where $[y]_t$ is the quadratic variation of (y_t) , then almost surely $\lim_{t \rightarrow \infty} \frac{y_t}{t} = 0$.

Consider the process $y_t(\mu_1, \mu_2)$ for a fixed $(\mu_1, \mu_2) \in \diamond$. By definition $y_t = \sum_{s=1}^t \varphi_s(\mu_1, \mu_2) - \bar{\varphi}_s(\mu_1, \mu_2)$. As established in the proof of Lemma **OA.5**, for every s , $\bar{\varphi}_s(\mu_1, \mu_2) = \mathbb{E}[\varphi_s(\mu_1, \mu_2)|\mathcal{F}_{s-1}]$.

So for $t' < t$,

$$\begin{aligned}
\mathbb{E}[y_t(\mu_1, \mu_2) | \mathcal{F}_{t'}] &= \sum_{s=1}^{t'} \varphi_s(\mu_1, \mu_2) - \bar{\varphi}_s(\mu_1, \mu_2) + \mathbb{E} \left[\sum_{s=t'+1}^t \varphi_s(\mu_1, \mu_2) - \bar{\varphi}_s(\mu_1, \mu_2) | \mathcal{F}_{t'} \right] \\
&= \sum_{s=1}^{t'} \varphi_s(\mu_1, \mu_2) - \bar{\varphi}_s(\mu_1, \mu_2) + \sum_{s=t'+1}^t \mathbb{E}[\mathbb{E}[\varphi_s(\mu_1, \mu_2) - \bar{\varphi}_s(\mu_1, \mu_2) | \mathcal{F}_{s-1}] | \mathcal{F}_{t'}] \\
&= \sum_{s=1}^{t'} \varphi_s(\mu_1, \mu_2) - \bar{\varphi}_s(\mu_1, \mu_2) + 0 \\
&= y_{t'}(\mu_1, \mu_2).
\end{aligned}$$

This shows $(y_t(\mu_1, \mu_2))_t$ is a martingale. Also,

$$\begin{aligned}
[y(\mu_1, \mu_2)]_t &= \sum_{s=1}^{t-1} \mathbb{E}[(y_s(\mu_1, \mu_2) - y_{s-1}(\mu_1, \mu_2))^2 | \mathcal{F}_{s-1}] \\
&= \sum_{s=1}^{t-1} \mathbb{E}[\xi_s^2(\mu_1, \mu_2) | \mathcal{F}_{s-1}] \\
&\leq \kappa_\xi \cdot t
\end{aligned}$$

by Lemma OA.5. Therefore [Heidhues, Koszegi, and Strack \(2018\)](#) Proposition 10 applies. \square

Lemma OA.7. $\lim_{t \rightarrow \infty} \sup_{(\mu_1, \mu_2) \in \diamond} \left| \frac{y_t(\mu_1, \mu_2)}{t} \right| = 0$ almost surely.

Proof. This argument is similar to Lemma 11 in [Heidhues, Koszegi, and Strack \(2018\)](#). I apply Lemma 2 of [Andrews \(1992\)](#), which says to prove this result I just need to check conditions BD, P-SSLN, and S-LIP from [Andrews \(1992\)](#). BD holds because \diamond is a bounded subset of \mathbb{R}^2 . P-SSLN holds because by Lemma OA.6, which shows for all $(\mu_1, \mu_2) \in \diamond$, $\lim_{t \rightarrow \infty} \left| \frac{y_t(\mu_1, \mu_2)}{t} \right| = 0$ almost surely.

Condition S-LIP is essentially a Lipschitz continuity condition. It requires finding sequence of random variables B_t such that $|\xi_t(\mu_1, \mu_2) - \xi_t(\mu'_1, \mu'_2)| \leq B_t \cdot (|\mu_1 - \mu'_1| + |\mu_2 - \mu'_2|)$ almost surely, such that these random variables satisfy

$$\sup_{t \geq 1} \frac{1}{t} \sum_{s=1}^t \mathbb{E}[B_s] < \infty, \text{ and } \lim_{t \rightarrow \infty} \frac{1}{t} \sum_{s=1}^t (B_s - \mathbb{E}[B_s]) = 0 \text{ almost surely.}$$

But for every ω ,

$$\begin{aligned}
|\xi_t(\mu_1, \mu_2) - \xi_t(\mu'_1, \mu'_2)| &\leq \mathbf{1}\{X_{1,s} \leq \tilde{C}_s\} \cdot \sigma^{-2} \cdot (|\mu_2 - \mu'_2| + \gamma|\mu_1 - \mu'_1|) \\
&\quad + \sigma^{-2} \mathbb{P}[X_1 \leq \tilde{C}_s] \cdot (|\mu_2 - \mu'_2| + \gamma|\mu_1 - \mu'_1|) \\
&\leq 2\sigma^{-2} (|\mu_2 - \mu'_2| + \gamma|\mu_1 - \mu'_1|).
\end{aligned}$$

Setting B_s as the constant $2\sigma^{-2}(1 + \gamma)$ for every s satisfies S-LIP. \square

OA 2.5 Bounds on Asymptotic Beliefs and Asymptotic Cutoffs

Recall that Lemma OA.2 implies that if we draw the line with slope $-\gamma$ through the point (μ_1^\bullet, μ_2) , all pairs of fundamentals on this line have the same optimal cutoff threshold. Then against any feasible model $\Psi(\mu_1, \mu_2; \gamma)$ with $(\mu_1, \mu_2) \in \diamond$, the best cutoff strategy is between $\underline{c}^\circ := C(\mu_1^\bullet, \underline{\mu}_2^\circ; \gamma)$ and $\bar{c}^\circ := C(\mu_1^\bullet, \bar{\mu}_2^\circ; \gamma)$.

For $\mu_2^l \leq \mu_2^h$ in the interval $[\underline{\mu}_2^\circ, \bar{\mu}_2^\circ]$, let $\diamond_{[\mu_2^l, \mu_2^h]} \subseteq \diamond$ be constructed from \diamond by translating its top and bottom edges towards the center, so that they pass through (μ_1^\bullet, μ_2^l) and (μ_1^\bullet, μ_2^h) respectively.

Lemma OA.8. *For $\underline{c} \geq \underline{c}^\circ$, if $\liminf_{t \rightarrow \infty} \tilde{C}_t \geq \underline{c}$ almost surely, then $\lim_{t \rightarrow \infty} \tilde{G}_t(\diamond_{[\mu_2^\circ, \mu_2^*(\underline{c})]}) = 0$ almost surely. Also, for $\bar{c} \leq \bar{c}^\circ$, if $\limsup_{t \rightarrow \infty} \tilde{C}_t \leq \bar{c}$ almost surely, then $\lim_{t \rightarrow \infty} \tilde{G}_t(\diamond_{(\mu_2^*(\bar{c}), \bar{\mu}_2^\circ]}) = 0$ almost surely.*

Proof. I first show that for all $\epsilon > 0$, there exists $\delta > 0$ such that almost surely,

$$\liminf_{t \rightarrow \infty} \inf_{(\mu_1, \mu_2) \in \diamond_{[\mu_2^\circ, \mu_2^*(\underline{c}) - \epsilon]}} \frac{1}{t} \frac{\partial \ell_t}{\partial \mu_2}(\mu_1, \mu_2) \geq \delta.$$

From Lemma OA.4, we may rewrite LHS as

$$\liminf_{t \rightarrow \infty} \inf_{(\mu_1, \mu_2) \in \diamond_{[\mu_2^\circ, \mu_2^*(\underline{c}) - \epsilon]}} \left[\frac{1}{t} \frac{D_2 g(\mu_1, \mu_2)}{g(\mu_1, \mu_2)} + \frac{y_t(\mu_1, \mu_2)}{t} + \frac{z_t(\mu_1, \mu_2)}{t} \right],$$

which is no smaller than taking the inf separately across the three terms in the bracket,

$$\begin{aligned} & \liminf_{t \rightarrow \infty} \inf_{(\mu_1, \mu_2) \in \diamond_{[\mu_2^\circ, \mu_2^*(\underline{c}) - \epsilon]}} \frac{1}{t} \frac{D_2 g(\mu_1, \mu_2)}{g(\mu_1, \mu_2)} + \liminf_{t \rightarrow \infty} \inf_{(\mu_1, \mu_2) \in \diamond_{[\mu_2^\circ, \mu_2^*(\underline{c}) - \epsilon]}} \frac{y_t(\mu_1, \mu_2)}{t} \\ & + \liminf_{t \rightarrow \infty} \inf_{(\mu_1, \mu_2) \in \diamond_{[\mu_2^\circ, \mu_2^*(\underline{c}) - \epsilon]}} \frac{z_t(\mu_1, \mu_2)}{t}. \end{aligned}$$

Since $D_2 g/g$ is bounded on \diamond as $D_2 g$ is continuous and g is continuous and strictly positive on the compact set \diamond , the first term is 0 for every ω . To deal with the second term,

$$\liminf_{t \rightarrow \infty} \inf_{(\mu_1, \mu_2) \in \diamond_{[\mu_2^\circ, \mu_2^*(\underline{c}) - \epsilon]}} \frac{y_t(\mu_1, \mu_2)}{t} \geq \liminf_{t \rightarrow \infty} \inf_{(\mu_1, \mu_2) \in \diamond} - \left| \frac{y_t(\mu_1, \mu_2)}{t} \right| = \liminf_{t \rightarrow \infty} \left\{ -1 \cdot \sup_{(\mu_1, \mu_2) \in \diamond} \left| \frac{y_t(\mu_1, \mu_2)}{t} \right| \right\}.$$

Lemma OA.7 gives $\lim_{t \rightarrow \infty} \sup_{(\mu_1, \mu_2) \in \diamond} \left| \frac{y_t(\mu_1, \mu_2)}{t} \right| = 0$ almost surely. Hence, we conclude

$$\liminf_{t \rightarrow \infty} \inf_{(\mu_1, \mu_2) \in \diamond_{[\mu_2^\circ, \mu_2^*(\underline{c}) - \epsilon]}} \frac{y_t(\mu_1, \mu_2)}{t} \geq 0$$

almost surely.

It suffices then to find $\delta > 0$ and show $\liminf_{t \rightarrow \infty} \inf_{(\mu_1, \mu_2) \in \diamond_{[\underline{\mu}_2^\circ, \mu_2^*(\underline{c}) - \epsilon]}} \frac{z_t(\mu_1, \mu_2)}{t} \geq \delta$ almost surely. Since z_t is the sum of $\bar{\varphi}_s$ terms that are decreasing functions of $\mu_2 + \gamma\mu_1$, the inner inf is always achieved at $(\mu_1, \mu_2) = (\mu_1^\bullet, \mu_2^*(\underline{c}) - \epsilon)$. So we get

$$\begin{aligned} \liminf_{t \rightarrow \infty} \inf_{(\mu_1, \mu_2) \in \diamond_{[\underline{\mu}_2^\circ, \mu_2^*(\underline{c}) - \epsilon]}} \frac{z_t(\mu_1, \mu_2)}{t} &= \liminf_{t \rightarrow \infty} \frac{z_t(\mu_1^\bullet, \mu_2^*(\underline{c}) - \epsilon)}{t} \\ &= \liminf_{t \rightarrow \infty} \frac{1}{t} \left[\sum_{s=1}^t \bar{\varphi}_s(\mu_1^\bullet, \mu_2^*(\underline{c}) - \epsilon) \right]. \end{aligned}$$

The definition of $\mu_2^*(\underline{c})$ is such that $\mu_2^\bullet - \mu_2^*(\underline{c}) - \gamma(\mu_1^\bullet - \mathbb{E}[X_1 | X_1 \leq \underline{c}]) = 0$. So for any $\tilde{c} \geq \underline{c}$, since $\gamma > 0$,

$$\begin{aligned} \mu_2^\bullet - \mu_2^*(\underline{c}) - \gamma(\mu_1^\bullet - \mathbb{E}[X_1 | X_1 \leq \tilde{c}]) &\geq 0 \\ \mu_2^\bullet - (\mu_2^*(\underline{c}) - \epsilon) - \gamma(\mu_1^\bullet - \mathbb{E}[X_1 | X_1 \leq \tilde{c}]) &\geq \epsilon. \end{aligned}$$

So at any $\tilde{c} \geq \underline{c}$,

$$\sigma^{-2} \mathbb{P}[X_1 \leq \tilde{c}] \cdot (\mu_2^\bullet - (\mu_2^*(\underline{c}) - \epsilon) - \gamma(\mu_1^\bullet - \mathbb{E}[X_1 | X_1 \leq \tilde{c}])) \geq \sigma^{-2} \mathbb{P}[X_1 \leq \underline{c}] \cdot \epsilon$$

Along any ω where $\liminf_{t \rightarrow \infty} \tilde{C}_t \geq \underline{c}$, we therefore have

$$\liminf_{t \rightarrow \infty} \bar{\varphi}_s(\mu_1^\bullet, \mu_2^*(\underline{c}) - \epsilon) \geq \sigma^{-2} \mathbb{P}[X_1 \leq \underline{c}] \cdot \epsilon.$$

Put $\delta = \sigma^{-2} \mathbb{P}[X_1 \leq \underline{c}] \cdot \epsilon$. This shows almost surely,

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \left[\sum_{s=1}^t \bar{\varphi}_s(\mu_1^\bullet, \mu_2^*(\underline{c}) - \epsilon) \right] \geq \delta.$$

From here, the same argument as in the proof of Lemma OA.3 shows $\lim_{t \rightarrow \infty} \tilde{G}_t(\diamond_{[\underline{\mu}_2^\circ, \mu_2^*(\underline{c}) - \epsilon]}) = 0$ almost surely. Since the choice of $\epsilon > 0$ is arbitrary, this establishes the first part of the lemma.

The proof of the second part of the statement is exactly symmetric. To sketch the argument, we need to show that for all $\epsilon > 0$, there exists $\delta > 0$ such that almost surely,

$$\limsup_{t \rightarrow \infty} \sup_{(\mu_1, \mu_2) \in \diamond_{[\mu_2^*(\bar{c}) + \epsilon, \bar{\mu}_2^\circ]}} \frac{1}{t} \frac{\partial \ell_t}{\partial \mu_2}(\mu_1, \mu_2) \leq -\delta.$$

This essentially reduces to analyzing

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \left[\sum_{s=1}^t \bar{\varphi}_s(\mu_1^\bullet, \mu_2^*(\bar{c}) + \epsilon) \right].$$

For any $\tilde{c} \leq \bar{c}$, since $\gamma > 0$,

$$\begin{aligned} \mu_2^\bullet - \mu_2^*(\bar{c}) - \gamma(\mu_1^\bullet - \mathbb{E}[X_1 | X_1 \leq \tilde{c}]) &\leq 0 \\ \mu_2^\bullet - (\mu_2^*(\bar{c}) + \epsilon) - \gamma(\mu_1^\bullet - \mathbb{E}[X_1 | X_1 \leq \tilde{c}]) &\leq -\epsilon. \end{aligned}$$

For every t and along every ω , $\tilde{C}_t(\omega) \geq \underline{c}^\circ$, as cutoffs below this value cannot be myopically optimal given any belief about second-period fundamental supported on \diamond . So along any ω such that $\limsup_{t \rightarrow \infty} \tilde{C}_t(\omega) \leq \bar{c}$, we have $\limsup_{t \rightarrow \infty} \bar{\varphi}_s(\mu_1^\bullet, \mu_2^*(\bar{c}) + \epsilon) \leq \sigma^{-2} \mathbb{P}[X_1 \leq \underline{c}^\circ] \cdot (-\epsilon)$. Setting $\delta := \sigma^{-2} \mathbb{P}[X_1 \leq \underline{c}^\circ] \cdot (\epsilon)$, we get $\limsup_{t \rightarrow \infty} \frac{1}{t} \left[\sum_{s=1}^t \bar{\varphi}_s(\mu_1^\bullet, \mu_2^*(\bar{c}) + \epsilon) \right] \leq -\delta$ almost surely. \square

Now, I use a bound on agents' asymptotic beliefs about μ_2 to deduce asymptotic restrictions on their cutoffs.

Lemma OA.9. *Suppose that there are $\underline{\mu}_2^\circ \leq \mu_2^l < \mu_2^h \leq \bar{\mu}_2^\circ$ such that $\lim_{t \rightarrow \infty} \tilde{G}_t(\diamond_{[\mu_2^l, \mu_2^h]}) = 1$ almost surely. Then $\liminf_{t \rightarrow \infty} \tilde{C}_t \geq C(\mu_1^\bullet, \mu_2^l; \gamma)$ and $\limsup_{t \rightarrow \infty} \tilde{C}_t \leq C(\mu_1^\bullet, \mu_2^h; \gamma)$ almost surely.*

Proof. I show $\liminf_{t \rightarrow \infty} \tilde{C}_t \geq C(\mu_1^\bullet, \mu_2^l; \gamma)$ almost surely. The argument establishing $\limsup_{t \rightarrow \infty} \tilde{C}_t \leq C(\mu_1^\bullet, \mu_2^h; \gamma)$ is symmetric.

Let $c^l = C(\mu_1^\bullet, \mu_2^l; \gamma)$, and recall before we defined $\underline{c}^\circ := C(\mu_1^\bullet, \underline{\mu}_2^\circ; \gamma)$ and $\bar{c}^\circ := C(\mu_1^\bullet, \bar{\mu}_2^\circ; \gamma)$.

For $(\mu_1^\bullet, \mu_2) \in \diamond$, let $L(\mu_2) \subseteq \diamond$ be the line segment in $\text{supp}(g)$ with slope $-\gamma$ that contains the point (μ_1^\bullet, μ_2) . By Lemma OA.2, $C(\mu_1', \mu_2'; \gamma) = C(\mu_1^\bullet, \mu_2; \gamma)$ for all $(\mu_1', \mu_2') \in L(\mu_2)$. Since $c \mapsto U(c; \mu_1, \mu_2)$ is single peaked for every (μ_1, μ_2) , and since $c^l \leq C(\mu_1^\bullet, \mu_2; \gamma)$ for all $\mu_2 \in [\mu_2^l, \mu_2^h]$, we also get $c^l \leq C(\mu_1', \mu_2'; \gamma)$ for every $(\mu_1', \mu_2') \in \diamond_{[\mu_2^l, \mu_2^h]}$, since $\diamond_{[\mu_2^l, \mu_2^h]}$ is the union of the line segments, $\diamond_{[\mu_2^l, \mu_2^h]} = \cup_{\mu_2 \in [\mu_2^l, \mu_2^h]} L(\mu_2)$.

Fix some $\epsilon > 0$. We get $U(c^l; \mu_1, \mu_2) - U(c^l - \epsilon; \mu_1, \mu_2) > 0$ for every $(\mu_1, \mu_2) \in \diamond_{[\mu_2^l, \mu_2^h]}$. As $(\mu_1, \mu_2) \mapsto (U(c^l; \mu_1, \mu_2) - U(c^l - \epsilon; \mu_1, \mu_2))$ is continuous, there exists some $\kappa^* > 0$ so that $U(c^l; \mu_1, \mu_2) - U(c^l - \epsilon; \mu_1, \mu_2) > \kappa^*$ for all $(\mu_1, \mu_2) \in \diamond_{[\mu_2^l, \mu_2^h]}$. In particular, if $\nu \in \Delta(\diamond_{[\mu_2^l, \mu_2^h]})$ is a belief about fundamentals, then $\int U(c^l; \mu_1, \mu_2) - U(c^l - \epsilon; \mu_1, \mu_2) > d\nu(\mu) > \kappa^*$.

Now, let

$$\begin{aligned} \bar{\kappa} &:= \sup_{c \in [\underline{c}^\circ, \bar{c}^\circ]} \sup_{(\mu_1, \mu_2) \in \diamond} U(c; \mu_1, \mu_2), \\ \underline{\kappa} &:= \inf_{c \in [\underline{c}^\circ, \bar{c}^\circ]} \inf_{(\mu_1, \mu_2) \in \diamond} U(c; \mu_1, \mu_2). \end{aligned}$$

Find $p \in (0, 1)$ so that $p\kappa^* - (1-p)(\bar{\kappa} - \underline{\kappa}) = 0$. At any belief $\hat{\nu} \in \Delta(\diamond)$ that assigns more than probability p to the sub-parallellogram $\diamond_{[\mu_2^l, \mu_2^h]}$, the optimal cutoff is larger than $c^l - \epsilon$. To see this, take any $\hat{c} \leq c^l - \epsilon$ and I will show \hat{c} is suboptimal. If $\hat{c} < \underline{c}$, then it is suboptimal after any belief on \diamond . If $\underline{c} \leq \hat{c} \leq c^l - \epsilon$, I show that

$$\int U(c^l; \mu_1, \mu_2) - U(\hat{c}; \mu_1, \mu_2) d\hat{\nu}(\mu) > 0.$$

To see this, we may decompose $\hat{\nu}$ as the mixture of a probability measure ν on $\diamond_{[\mu_2^l, \mu_2^h]}$ and another probability measure ν^c on $\diamond \setminus \diamond_{[\mu_2^l, \mu_2^h]}$. Let $\hat{p} > p$ be the probability that ν assigns to $\diamond_{[\mu_2^l, \mu_2^h]}$. The above integral is equal to:

$$\hat{p} \int_{\diamond_{[\mu_2^l, \mu_2^h]}} U(c^l; \mu_1, \mu_2) - U(\hat{c}; \mu_1, \mu_2) d\nu(\mu) + (1 - \hat{p}) \int_{\diamond \setminus \diamond_{[\mu_2^l, \mu_2^h]}} U(c^l; \mu_1, \mu_2) - U(\hat{c}; \mu_1, \mu_2) d\nu^c(\mu)$$

Since c^l is to the left of the optimal cutoff for all $(\mu_1, \mu_2) \in \diamond_{[\mu_2^l, \mu_2^h]}$ and $\hat{c} \leq c^l - \epsilon$, then $U(\hat{c}; \mu_1, \mu_2) \leq U(c^l - \epsilon; \mu_1, \mu_2)$ for all $(\mu_1, \mu_2) \in \diamond_{[\mu_2^l, \mu_2^h]}$. The first summand is no less than

$$\hat{p} \int_{\diamond_{[\mu_2^l, \mu_2^h]}} U(c^l; \mu_1, \mu_2) - U(c^l - \epsilon; \mu_1, \mu_2) d\nu(\mu) \geq \hat{p}\kappa^*.$$

Also, the integrand in the second summand is no smaller than $-(\bar{\kappa} - \underline{\kappa})$, therefore $\int U(c^l; \mu_1, \mu_2) - U(\hat{c}; \mu_1, \mu_2) d\hat{\nu}(\mu) \geq \hat{p}\kappa^* - (1 - \hat{p})(\bar{\kappa} - \underline{\kappa})$. Since $\hat{p} > p$, we get $\hat{p}\kappa^* - (1 - \hat{p})(\bar{\kappa} - \underline{\kappa}) > 0$ as desired.

Along any sample path ω where $\lim_{t \rightarrow \infty} \tilde{G}_t(\diamond_{[\mu_2^l, \mu_2^h]})(\omega) = 1$, eventually $\tilde{G}_t(\diamond_{[\mu_2^l, \mu_2^h]})(\omega) > p$ for all large enough t , meaning $\liminf_{t \rightarrow \infty} \tilde{C}_t(\omega) \geq c^l - \epsilon$. Since $\lim_{t \rightarrow \infty} \tilde{G}_t(\diamond_{[\mu_2^l, \mu_2^h]})(\omega) = 1$ almost surely, this shows $\liminf_{t \rightarrow \infty} \tilde{C}_t \geq C(\mu_1^\bullet, \mu_2^l; \gamma) - \epsilon$ almost surely. Since the choice of $\epsilon > 0$ was arbitrary, we in fact conclude $\liminf_{t \rightarrow \infty} \tilde{C}_t \geq C(\mu_1^\bullet, \mu_2^l; \gamma)$ almost surely. \square

OA 2.6 The Contraction Map

I now combine the results established so far to prove the convergence statement in Theorem 1.

Proof. Let $\mu_{2,[1]}^l := \mu_2^\circ$, $\mu_{2,[1]}^h := \bar{\mu}_2^\circ$. For $k = 2, 3, \dots$, iteratively define $\mu_{2,[k]}^l := \mathcal{I}(\mu_{2,[k-1]}^l; \gamma)$ and $\mu_{2,[k]}^h := \mathcal{I}(\mu_{2,[k-1]}^h; \gamma)$.

From Lemma OA.9, if $\lim_{t \rightarrow \infty} \tilde{G}_t(\diamond_{[\mu_{2,[k]}^l, \mu_{2,[k]}^h]}) = 1$ almost surely, then $\liminf_{t \rightarrow \infty} \tilde{C}_t \geq C(\mu_1^\bullet, \mu_{2,[k]}^l; \gamma)$ and $\limsup_{t \rightarrow \infty} \tilde{C}_t \leq C(\mu_1^\bullet, \mu_{2,[k]}^h; \gamma)$ almost surely. But using these conclusions

in Lemma OA.8, we further deduce that

$$\lim_{t \rightarrow \infty} \tilde{G}_t(\diamond_{[\mu_2^*(C(\mu_1^\bullet, \mu_{2,[k]}^l; \gamma)), \mu_2^*(C(\mu_1^\bullet, \mu_{2,[k]}^h; \gamma))]) = 1$$

almost surely, that is to say $\lim_{t \rightarrow \infty} \tilde{G}_t(\diamond_{[\mu_{2,[k+1]}^l, \mu_{2,[k+1]}^h]}) = 1$ almost surely.

The iterates $(\mu_{2,[k]}^l)_{k \geq 1}$ and $(\mu_{2,[k]}^h)_{k \geq 1}$ are the iterates of a contraction map, so $\lim_{k \rightarrow \infty} \mu_{2,[k]}^l = \mu_2^\bullet = \lim_{k \rightarrow \infty} \mu_{2,[k]}^h$. Thus, agent's posterior converges in L^1 to the line segment with slope $-\gamma$ containing μ_2^∞ almost surely (since the support of the prior is bounded).

In addition, the sequences of bounds on asymptotic actions also converge by continuity, $\lim_{k \rightarrow \infty} C(\mu_1^\bullet, \mu_{2,[k]}^l; \gamma) = c^\infty = \lim_{k \rightarrow \infty} C(\mu_1^\bullet, \mu_{2,[k]}^h; \gamma)$. This implies $\lim_{t \rightarrow \infty} \tilde{C}_t = c^\infty$ almost surely.

Finally, combining the asymptotic belief result with Lemma OA.3, we see that in fact \tilde{G}_t converges in L_1 to the point $(\mu_1^\bullet, \mu_2^\infty)$ almost surely. \square

OA 3 Additional Extensions

OA 3.1 Draws as Costs

In the baseline model, I have studied optimal-stopping problems satisfying Assumption 1. One implication of Assumption 1 is that higher draws are more beneficial to the agent, as u_1 and u_2 are strictly increasing functions of the draws in their respective periods. In this section, I verify the robustness of my positive-feedback loop result when draws are interpreted as costs. Here is the canonical example to keep in mind:

Example OA.1 (Do It Now or Later). The agent has two periods to complete a task. In period 1, she draws her cost of completing the task today, $X_1 = x_1$. The agent chooses between paying x_1 and finishing the task, or waiting until period 2. If she decides to wait, she will draw another cost $X_2 = x_2$ in period 2, which she must then pay. So, $u_1(x_1) = -x_1$ and $u_2(x_1, x_2) = -x_2$.

The purpose of this robustness check is to show that the positive-feedback loop result does not rely on an asymmetric assumption about the optimal-stopping problem. The idea is that in optimal-stopping problems like Example OA.1, the subjectively optimal stopping rule given any beliefs about the fundamentals in the two periods features stopping for low values of X_1 . This means agents observed censored datasets from their predecessors where X_2 is only observed following high values of X_1 , the “opposite” kind of endogenous censoring compared to what happens in problems satisfying Assumption 1. Now, a more heavily censored dataset induces a higher belief about the second-period mean in the next generation, which causes

the next generation to accept higher costs in the first period. This exacerbates the censoring and the positive feedback cycle again obtains.

More generally, I will consider payoff functions $u_1(x_1)$, $u_2(x_1, x_2)$ satisfy the following assumptions.

Assumption OA.1. *The payoff functions satisfy:*

- (a) For $x'_1 > x''_1$ and $x'_2 > x''_2$, $u_1(x'_1) < u_1(x''_1)$ and $u_2(x'_1, x'_2) < u_2(x''_1, x''_2)$.
- (b) For $x'_1 > x''_1$ and any \bar{x}_2 , $u_1(x'_1) - u_1(x''_1) < -|u_2(x'_1, \bar{x}_2) - u_2(x''_1, \bar{x}_2)|$.
- (c) There exist $x_1^h, x_2^l, x_1^l, x_2^h \in \mathbb{R}$ so that $u_1(x_1^h) - u_2(x_1^h, x_2^l) < 0$, while $u_1(x_1^l) - u_2(x_1^l, x_2^h) > 0$
- (d) u_1, u_2 are continuous. Also, for any $\bar{x}_1 \in \mathbb{R}$, $x_2 \mapsto u_2(\bar{x}_1, x_2)$ is absolutely integrable with respect to any Gaussian distribution on \mathbb{R} .

I show that the subjectively optimal stopping strategy under dogmatic belief in fundamentals μ_1, μ_2 takes a cutoff form, but the agent stops in period 1 for low realizations of period 1 costs, $X_1 \leq c$. Furthermore, the optimal cutoff increases in μ_2 .

Proposition OA.11. *Under Assumption OA.1 and for $\gamma > 0$,*

- Under each subjective model $\Psi(\mu_1, \mu_2; \gamma)$, there exists a cutoff threshold $C^{\text{cost}}(\mu_1, \mu_2; \gamma) \in \mathbb{R}$ such that it is strictly optimal to continue whenever $x_1 > C^{\text{cost}}(\mu_1, \mu_2; \gamma)$ and strictly optimal to stop whenever $x_1 < C^{\text{cost}}(\mu_1, \mu_2; \gamma)$.
- For every $\mu_1 \in \mathbb{R}$, $\mu_2 \mapsto C^{\text{cost}}(\mu_1, \mu_2; \gamma)$ is strictly increasing.

Proof. Consider the pair of payoff functions $\tilde{u}_1 : \mathbb{R} \rightarrow \mathbb{R}$ and $\tilde{u}_2 : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by $\tilde{u}_1(\tilde{x}_1) := u_1(-\tilde{x}_1)$ and $\tilde{u}_2(\tilde{x}_1, \tilde{x}_2) := u_2(-\tilde{x}_1, -\tilde{x}_2)$. It is easy to verify that since u_1, u_2 satisfy Assumption OA.1, \tilde{u}_1, \tilde{u}_2 must satisfy Assumption 1.

When $(X_1, X_2) \sim \Psi(\mu_1, \mu_2; \gamma)$, we also have $(\tilde{X}_1, \tilde{X}_2) \sim \Psi(-\mu_1, -\mu_2; \gamma)$, where $(\tilde{X}_1, \tilde{X}_2) = (-X_1, -X_2)$. The best stopping strategy under the payoff functions \tilde{u}_1, \tilde{u}_2 when draws are generated from $\Psi(-\mu_1, -\mu_2; \gamma)$ involves the cutoff threshold $C(-\mu_1, -\mu_2; \gamma)$, stopping whenever \tilde{X}_1 exceeds the threshold and continuing whenever \tilde{X}_1 falls below it. Here, $C(-\mu_1, -\mu_2; \gamma)$ is the usual acceptance threshold from Proposition 1.

By the relationship between u_1, u_2 and \tilde{u}_1, \tilde{u}_2 , we deduce that the optimal stopping strategy under the payoff functions u_1, u_2 when draws are generated from $\Psi(\mu_1, \mu_2; \gamma)$ involves the cutoff threshold $C^{\text{cost}}(\mu_1, \mu_2; \gamma) := -C(-\mu_1, -\mu_2; \gamma)$. The agent should stop when the first (cost-based) draw falls below $C^{\text{cost}}(\mu_1, \mu_2; \gamma)$, and continue when the first draw exceeds the cutoff.

For $\mu'_2 > \mu_2$, we have $C(-\mu_1, -\mu'_2; \gamma) < C(-\mu_1, -\mu_2; \gamma)$ by Proposition 1. So,

$$C^{\text{cost}}(\mu_1, \mu'_2; \gamma) = -1 \cdot C(-\mu_1, -\mu'_2; \gamma) > -1 \cdot C(-\mu_1, -\mu_2; \gamma) = C^{\text{cost}}(\mu_1, \mu_2; \gamma)$$

as desired. \square

As Proposition OA.11 shows, the subjectively optimal stopping rules in problems satisfying Assumption OA.1 imply a different kind of censoring than in the baseline model. Specifically, the history contains the second-period draw only when X_1 is high. For $c \in \mathbb{R}$, let \bar{S}_c denote the stopping strategy $S(x_1) = \text{Continue}$ if $x_1 > c$, $S(x_1) = \text{Stop}$ if $x_1 \leq c$. The bar notation distinguishes it from S_c , the stopping strategy with the stopping region $[c, \infty)$. For $c, \mu_1, \mu_2 \in \mathbb{R}$, the KL divergence between $\mathcal{H}(\Psi^\bullet; \bar{S}_c)$ and $\mathcal{H}(\Psi(\mu_1, \mu_2; \gamma); \bar{S}_c)$ is given by

$$\begin{aligned} & \int_{-\infty}^c \phi(x_1; \mu_1^\bullet, \sigma^2) \cdot \ln \left(\frac{\phi(x_1; \mu_1^\bullet, \sigma^2)}{\phi(x_1; \mu_1, \sigma^2)} \right) dx_1 \\ & + \int_c^\infty \left\{ \int_{-\infty}^\infty \phi(x_1; \mu_1^\bullet, \sigma^2) \cdot \phi(x_2; \mu_2^\bullet, \sigma^2) \cdot \ln \left[\frac{\phi(x_1; \mu_1^\bullet, \sigma^2) \cdot \phi(x_2; \mu_2^\bullet, \sigma^2)}{\phi(x_1; \mu_1, \sigma^2) \cdot \phi(x_2; \mu_2 - \gamma(x_1 - \mu_1), \sigma^2)} \right] dx_2 \right\} dx_1. \end{aligned}$$

Proposition OA.12. *The pseudo-true fundamentals minimizing $D_{KL}(\mathcal{H}(\Psi^\bullet; \bar{S}_c) \parallel \mathcal{H}(\Psi(\mu_1, \mu_2; \gamma); \bar{S}_c))$ are $\mu_1^*(c) = \mu_1^\bullet$ and*

$$\mu_2^*(c) = \mu_2^\bullet - \gamma (\mu_1^\bullet - \mathbb{E}[X_1 \mid X_1 \geq c]).$$

So $\mu_2^*(c)$ is strictly increasing in c .

Since $\mathbb{E}[X_1 \mid X_1 \geq c] > \mu_1^\bullet$ for every $c \in \mathbb{R}$ and $\gamma > 0$, this shows the pseudo-true second-period fundamental is always too high for every stopping strategy \bar{S}_c . Since high values of draws are bad news in the environment where draws are interpreted as costs, this shows the over-pessimism result from the baseline environment extends into this setting.

Proof. Rewrite $D_{KL}(\mathcal{H}(\Psi^\bullet; \bar{S}_c) \parallel \mathcal{H}(\Psi(\mu_1, \mu_2; \gamma); \bar{S}_c))$ as

$$\begin{aligned} & \int_{-\infty}^\infty \phi(x_1; \mu_1^\bullet, \sigma^2) \cdot \ln \left(\frac{\phi(x_1; \mu_1^\bullet, \sigma^2)}{\phi(x_1; \mu_1, \sigma^2)} \right) dx_1 \\ & + \int_c^\infty \phi(x_1; \mu_1^\bullet, \sigma^2) \cdot \int_{-\infty}^\infty \phi(x_2; \mu_2^\bullet, \sigma^2) \ln \left[\frac{\phi(x_2; \mu_2^\bullet, \sigma^2)}{\phi(x_2; \mu_2 - \gamma(x_1 - \mu_1), \sigma^2)} \right] dx_2 dx_1. \end{aligned}$$

The KL divergence between $\mathcal{N}(\mu_{\text{true}}, \sigma_{\text{true}}^2)$ and $\mathcal{N}(\mu_{\text{model}}, \sigma_{\text{model}}^2)$ is $\ln \frac{\sigma_{\text{model}}}{\sigma_{\text{true}}} + \frac{\sigma_{\text{true}}^2 + (\mu_{\text{true}} - \mu_{\text{model}})^2}{2\sigma_{\text{model}}^2} - \frac{1}{2}$, so we may simplify the first term and the inner integral of the second term.

$$\frac{(\mu_1 - \mu_1^\bullet)^2}{2\sigma^2} + \int_c^\infty \phi(x_1; \mu_1^\bullet, \sigma^2) \cdot \left[\frac{\sigma^2 + (\mu_2 - \gamma(x_1 - \mu_1) - \mu_2^\bullet)^2}{2\sigma^2} - \frac{1}{2} \right] dx_1.$$

Dropping constant terms not depending on μ_1 and μ_2 and multiplying by σ^2 , we get a simplified expression of the objective,

$$\xi(\mu_1, \mu_2) := \frac{(\mu_1 - \mu_1^\bullet)^2}{2} + \int_{-\infty}^c \phi(x_1; \mu_1^\bullet, \sigma^2) \cdot \left[\frac{(\mu_2 - \gamma(x_1 - \mu_1) - \mu_2^\bullet)^2}{2} \right] dx_1.$$

We have the partial derivatives by differentiating under the integral sign,

$$\begin{aligned} \frac{\partial \xi}{\partial \mu_2} &= \int_c^\infty \phi(x_1; \mu_1^\bullet, \sigma^2) \cdot (\mu_2 - \gamma(x_1 - \mu_1) - \mu_2^\bullet) dx_1 \\ \frac{\partial \xi}{\partial \mu_1} &= (\mu_1 - \mu_1^\bullet) + \gamma \int_c^\infty \phi(x_1; \mu_1^\bullet, \sigma^2) \cdot (\mu_2 - \gamma(x_1 - \mu_1) - \mu_2^\bullet) dx_1 \\ &= (\mu_1 - \mu_1^\bullet) + \gamma \frac{\partial \xi}{\partial \mu_2} \end{aligned}$$

By the first order conditions, at the minimum (μ_1^*, μ_2^*) , we must have:

$$\frac{\partial \xi}{\partial \mu_2}(\mu_1^*, \mu_2^*) = \frac{\partial \xi}{\partial \mu_1}(\mu_1^*, \mu_2^*) = 0 \Rightarrow \mu_1^* = \mu_1^\bullet.$$

So μ_2^* satisfies $\frac{\partial \xi}{\partial \mu_2}(\mu_1^\bullet, \mu_2^*) = 0$, which by straightforward algebra shows $\mu_2^*(c) = \mu_2^\bullet - \gamma(\mu_1^\bullet - \mathbb{E}[X_1 | X_1 \geq c])$. \square

Combining Propositions [OA.11](#) and [OA.12](#), we see that the positive-feedback loop between distorted actions and distorted beliefs is preserved when draws are interpreted as costs. Indeed, a higher belief about the second-period fundamental increases the threshold c , which leads to more severe censoring of the dataset as X_2 is only observed when $X_1 \geq c$. This more severely censored dataset, in turn, leads to even higher belief in the second-period fundamental by Proposition [OA.12](#). So as in Theorem [2](#), cutoff thresholds and beliefs about the fundamentals form monotonic sequences across generations $t \geq 1$.

OA 3.2 Population with Heterogeneity in Selection Neglect

I believe my learning environment is unlikely to evoke selection neglect, a psychology most likely to be present when the observed dataset contains does not contain reminders about selection. Censoring is highly explicit and salient in my setting, which is not the type of framing that typically evokes selection neglect.

In [Enke \(2017\)](#)'s experiment on selection neglect, players (one human subject and five computer players following a mechanical rule) are asked to guess a ‘‘state of the world’’ based on the average of 6 private signals. Players are sorted into one of two groups based on

whether their own private signal is high or low, then observe the signals of others in their group. In the baseline treatment, there is no reminder of the excluded data on the decision screen where subjects are shown the signals of others in the same group and asked to enter a guess. This treatment finds selection neglect. Another treatment where subjects are given a simple hint stating: “Also think about the computer players whom you do not communicate with!” reduces the number of selection neglecters by 60%. So I believe the much clearer reminders of selection in my environment should reduce the frequency of selection neglect even further.

Jehiel (2018) studies misperceived investment returns under selection neglect. In his model, each predecessor has a potential project and observes a private signal about the project’s quality. Predecessors with high signals implement their projects. Agents in the current generation observe the pool of implemented projects, then generate their own signals about the qualities of these observed projects. These signals are independent of the actual private signals that the predecessors used for implementation decisions. Current agents infer the conditional quality given each signal using the empirical mean quality among past implemented projects generating the same signal. This is another environment where the dataset contains no hints about the existence of excluded data (the unimplemented projects) or the selection criterion (the private signals of predecessors). In fact, if datasets in Jehiel (2018)’s setting record the complete experience of the predecessors in their decision problems, as is the case in my history datasets, then the misinference result no longer holds.

Nevertheless, in this section, I study an extension of the baseline model where a fraction $0 \leq \alpha < 1$ of agents in each generation has full selection neglect, while the remainder are baseline agents with the gambler’s fallacy. This mixture specification is inspired by Enke (2017)’s experimental results, who finds heterogeneity in subjects’ degree of selection neglect with the full-selection-neglect subjects and no-selection-neglect subjects together accounting for a majority of the population. On the other hand, Enke (2017) does not find a significant mass of subjects at any “intermediate” level of selection neglect.

To model agents with full selection neglect, I assume that when faced with a dataset of histories $(h_{1,n}, h_{2,n})_{n \in [0,1]}$, they treat $(h_{1,n})_{n \in [0,1]}$ as a sample from the unconditional distribution of X_1 , and $(h_{2,n})_{n: h_{2,n} \neq \emptyset}$ as an independent sample from the unconditional distribution of X_2 . Relative to the base line agents, they make the error of of the selection process behind which $h_{2,n}$ appear in the dataset: they are not censored at random, but only censored when $h_{1,n}$ exceeds the acceptance threshold used by the predecessors. In this environment, the gambler’s fallacy and selection neglect exactly cancel each other out, since in large datasets the mean of $h_{1,n}$ is μ_1^\bullet and the mean of uncensored $h_{2,n}$ is μ_2^\bullet . This shows that from the dataset $\mathcal{H}^\bullet(c)$ for any $c \in \mathbb{R}$, the selection neglecters correctly infer the fundamentals and

choose the stopping strategy with cutoff²⁶ $C(\mu_1^\bullet, \mu_2^\bullet; \gamma)$.

Now consider a baseline agent with the gambler's fallacy, facing a dataset of histories generated by a heterogeneous population of predecessors. A fraction α of the histories are generated by selection neglecters using the stopping strategy $S_{C(\mu_1^\bullet, \mu_2^\bullet; \gamma)}$. The remaining $1 - \alpha$ fraction are generated by baseline predecessors using the stopping strategy S_c . The next Proposition characterizes the pseudo-true fundamentals maximizing the weighted-average KL-divergence objective,

$$\alpha D_{KL}(\mathcal{H}^\bullet(C(\mu_1^\bullet, \mu_2^\bullet; \gamma)) \parallel \mathcal{H}(\Psi(\mu_1, \mu_2; \gamma); C(\mu_1^\bullet, \mu_2^\bullet; \gamma))) + (1 - \alpha) D_{KL}(\mathcal{H}^\bullet(c) \parallel \mathcal{H}(\Psi(\mu_1, \mu_2; \gamma); c)). \quad (4)$$

The proof is similar to that of Proposition 2, except replacing multiple previous generations with two sub-populations within the immediate predecessor generation, and weighing these sub-populations differently due to their relative sizes.

Proposition OA.13. *The pseudo-true fundamentals minimizing Equation (4) when baseline predecessors use the stopping threshold c is $\mu_1^{SN} = \mu_1^\bullet$,*

$$\begin{aligned} \mu_2^{SN}(c) = & \frac{\alpha \mathbb{P}[X_1 \leq C(\mu_1^\bullet, \mu_2^\bullet; \gamma)]}{\alpha \mathbb{P}[X_1 \leq C(\mu_1^\bullet, \mu_2^\bullet; \gamma)] + (1 - \alpha) \mathbb{P}[X_1 \leq c]} \cdot \mu_2^*(C(\mu_1^\bullet, \mu_2^\bullet; \gamma)) \\ & + \frac{(1 - \alpha) \mathbb{P}[X_1 \leq c]}{\alpha \mathbb{P}[X_1 \leq C(\mu_1^\bullet, \mu_2^\bullet; \gamma)] + (1 - \alpha) \mathbb{P}[X_1 \leq c]} \cdot \mu_2^*(c). \end{aligned}$$

Proof. Let $w_1 = \alpha, w_2 = 1 - \alpha, c_1 = C(\mu_1^\bullet, \mu_2^\bullet; \gamma), c_2 = c$. By the same argument as in the proof of Proposition 2, we may rewrite the weighted KL divergence as

$$\frac{(\mu_1 - \mu_1^\bullet)^2}{2\sigma^2} + \sum_{k=1}^2 w_k \left\{ \int_{-\infty}^{c_k} \phi(x_1; \mu_1^\bullet, \sigma^2) \cdot \left[\frac{\sigma^2 + (\mu_2 - \gamma(x_1 - \mu_1) - \mu_2^\bullet)^2}{2\sigma^2} - \frac{1}{2} \right] dx_1 \right\}.$$

Dropping terms not dependent on μ_1, μ_2 and multiplying through by σ^2 , we get the simplified objective

$$\xi^{SN}(\mu_1, \mu_2) := \frac{(\mu_1 - \mu_1^\bullet)^2}{2} + \sum_{k=1}^2 w_k \left\{ \int_{-\infty}^{c_k} \phi(x_1; \mu_1^\bullet, \sigma^2) \cdot \left[\frac{(\mu_2 - \gamma(x_1 - \mu_1) - \mu_2^\bullet)^2}{2\sigma^2} \right] dx_1 \right\}.$$

The same argument as in the proof of Proposition 2 shows that the first-order condition is

²⁶This cutoff may nevertheless differ from the objectively optimal one, since the selection neglecters also suffer from the gambler's fallacy, so they believe in the joint distribution $\Psi(\mu_1^\bullet, \mu_2^\bullet; \gamma)$.

only satisfied at $\mu_1^{SN} = \mu_1^\bullet$,

$$\mu_2^{SN} = \frac{1}{w_1 \mathbb{P}[X_1 \leq c_1] + w_2 \mathbb{P}[X_1 \leq c_2]} \sum_{k=1}^2 w_k \mathbb{P}[X_1 \leq c_k] \{ \mu_2^\bullet - \gamma (\mu_1^\bullet - \mathbb{E}[X_1 | X_1 \leq c_k]) \}.$$

This shows, in terms of expressions for pseudo-true fundamentals in the baseline model μ_2^* ,

$$\begin{aligned} \mu_2^{SN}(c) &= \frac{\alpha \mathbb{P}[X_1 \leq C(\mu_1^\bullet, \mu_2^\bullet; \gamma)]}{\alpha \mathbb{P}[X_1 \leq C(\mu_1^\bullet, \mu_2^\bullet; \gamma)] + (1 - \alpha) \mathbb{P}[X_1 \leq c]} \cdot \mu_2^*(C(\mu_1^\bullet, \mu_2^\bullet; \gamma)) \\ &\quad + \frac{(1 - \alpha) \mathbb{P}[X_1 \leq c]}{\alpha \mathbb{P}[X_1 \leq C(\mu_1^\bullet, \mu_2^\bullet; \gamma)] + (1 - \alpha) \mathbb{P}[X_1 \leq c]} \cdot \mu_2^*(c). \end{aligned}$$

□

That is, with a mixture of selection-neglecter and baseline predecessors, baseline agents' inference about the second-period fundamental is a convex combination between what they would infer from the histories of the selection neglecters alone and what they would infer from the histories of the baseline predecessors alone. The relative weights given to these two pseudo-true second-period fundamentals depend on the relative sizes of the two subpopulations, as well as on how frequently second-period draws are observed in each of the two sub-datasets.

Since both $\mu_2^*(C(\mu_1^\bullet, \mu_2^\bullet; \gamma))$ and $\mu_2^*(c)$ are strictly below μ_2^\bullet , we immediately conclude the same holds for $\mu_2^{SN}(c)$ for any $c \in \mathbb{R}$. This shows the robustness of the over-pessimism result from the main text to the presence of a fraction of selection neglecters.

Next, I compare a baseline society with no selection neglecters with a second society containing a positive fraction of selection neglecters. I show that when two societies start with the same generation 0 behavior, society with selection neglecters hold more optimistic beliefs about the second-period fundamental and use a higher stopping threshold in every generation $t \geq 1$. This is not simply due to the mechanical reason that the selection neglecters always make the correct inferences about the fundamentals, thereby making the "average" belief in the society more optimistic. The presence of the selection neglecters also moderates the over-pessimism of the baseline gambler's fallacy agents (without completely eliminating it), by making the censoring effect less severe.

Corollary A.3. *Let $0 < \alpha < 1$. Consider two societies, A and B, where society A has no selection neglecters and society B has an α fraction of selection neglecters in each generation. Suppose both societies start at the same initial condition $c_{[0]} \in \mathbb{R}$. For $t \geq 1$, consider the auxiliary learning environment and denote the baseline agents' beliefs and cutoff thresholds in society $k \in \{A, B\}$ as $\mu_{1,[t]}^k, \mu_{2,[t]}^k, c_{[t]}^k$. Then for every $t \geq 2$, $\mu_{2,[t]}^2 > \mu_{2,[t]}^1$ and $c_{[t]}^2 > c_{[t]}^1$.*

Proof. From Proposition OA.13 (and Proposition 2 for the case of $t = 1$), $\mu_{1,[t]}^A = \mu_{1,[t]}^B = \mu_1^\bullet$

for every $t \geq 1$. Also, in the first generation, $\mu_{2,[1]}^A = \mu_{2,[1]}^B$ and $c_{[1]}^A = c_{[1]}^B$ since both societies face the same dataset $\mathcal{H}^\bullet(c_{[0]})$. Since $\mu_{2,[1]}^A < \mu_2^\bullet$, we must have $c_{[1]}^A = C(\mu_1^\bullet, \mu_{2,[1]}^A; \gamma) < C(\mu_1^\bullet, \mu_2^\bullet; \gamma)$ by Lemma 1. In the second generation, $\mu_{2,[2]}^A = \mu_2^*(c_{[1]}^A)$ and $\mu_{2,[2]}^B$ is a convex combination between $\mu_2^*(c_{[1]}^B)$ and $\mu_2^*(C(\mu_1^\bullet, \mu_2^\bullet; \gamma))$. As $\mu_2^*(c_{[1]}^A) = \mu_2^*(c_{[1]}^B) < \mu_2^*(C(\mu_1^\bullet, \mu_2^\bullet; \gamma))$ due to Proposition 2, we conclude $\mu_{2,[2]}^B > \mu_{2,[2]}^A$ and hence $c_{[2]}^B > c_{[2]}^A$. But when $c_{[t]}^B > c_{[t]}^A$ and $C(\mu_1^\bullet, \mu_2^\bullet; \gamma) > c_{[t]}^A$ we have $\mu_2^*(c_{[t]}^A) < \mu_2^*(c_{[t]}^B)$, which shows in the next generation, $\mu_{2,[t+1]}^B$ is the convex combination of two terms both exceeding $\mu_2^*(c_{[t]}^A)$. This implies $\mu_{2,[t+1]}^B > \mu_{2,[t+1]}^A$ and $c_{[t+1]}^B > c_{[t+1]}^A$. By induction, the corollary holds for all $t \geq 2$. \square

OA 3.3 Misattribution of Reference Dependence

In this extension, I enrich the learning environment with misattribution of reference dependence, as studied in [Bushong and Gagnon-Bartsch \(2018\)](#). Briefly, the motivation is that individuals derive utility not only from the intrinsic value of an outcome, but also from how that outcome compares with their expectations. Failing to disentangle these two components of utility in recalling past experience may lead to mislearning. I consider a stylized model of misattribution of reference dependence where datasets do not record the draws from previous generation's decision problems, but rather the sum of the draws and a reference-dependent term. Agents in the current generation, however, act as if they face the usual dataset recording the unmodified decision histories.

To introduce the model formally, suppose predecessors believe in the subjective model $(X_1, X_2) \sim \Psi(\mu_1^\circ, \mu_2^\circ, \sigma^2, \sigma^2; \gamma)$. As I will assume agents have dogmatic beliefs in the correct variances, I will abbreviate this model simply as $(X_1, X_2) \sim \Psi(\mu_1^\circ, \mu_2^\circ; \gamma)$ for the remainder of this section. Fixing $\eta \geq 0$, the degree of reference dependence, let $Y_1 = X_1 + \eta(X_1 - \mu_1^\circ) = (1 + \eta)X_1 - \eta\mu_1^\circ$. Here, Y_1 represents a combination of the realization of first-period draw, plus an elation or disappointment term relative to the predecessor's expectation, μ_1° . Each predecessor records Y_1 instead of X_1 for his decision problem. Similarly, let

$$Y_2|(X_1 = x_1) = X_2 + \eta(X_2 - (\mu_2^\circ - \gamma(x_1 - \mu_1^\circ))).$$

Under the gambler's fallacy reasoning, a predecessor's expectation of the second draw depends not only on his beliefs about the fundamentals, $(\mu_1^\circ, \mu_2^\circ)$, but also on the realization of the first period draw in his decision problem. For a fixed value of the actual second draw X_2 , the encoded value Y_2 decreases in μ_2° and μ_1° , increases in X_1 . We have Y_2 decreasing in μ_2° because higher μ_2° raises expectation of second-period draw. Also, under the gambler's fallacy psychology, the predecessor's expectation of second-period draw decreases in $X_1 - \mu_1^\circ$, the first-period surprise.

An important difference from [Bushong and Gagnon-Bartsch \(2018\)](#)'s work is that I do

not include a loss-aversion term that asymmetrically distorts draws above and below expectations, leading to a greater extent of miscoding for disappointing outcomes. As I will show, this symmetric model of reference dependence will nevertheless produce asymmetric predictions. This is because the subjectively optimal stopping rule features a cutoff threshold, which endogenously leads to asymmetric censoring of data in my learning environment.

The key driver is mislearning is that agents entirely neglect the effects of reference dependence. This manifests in two ways. First, agents think of a censored dataset containing (Y_1, Y_2) pairs as a dataset containing (X_1, X_2) pairs, not knowing that the entries have been contaminated by the reference dependence terms. I assume that when predecessors stop according to the threshold $X_1 \geq c$, the next generation misperceives this stopping threshold as $(1 + \eta)c - \eta\mu_1^\circ$. This is because they face an infinite dataset of (Y_1, Y_2) pairs with Y_2 censored whenever $X_1 \geq c$, or equivalently $Y_1 \geq (1 + \eta)c - \eta\mu_1^\circ$. Second, agents do not take reference dependence into account in formulating their stopping strategy. That is, they observe the dataset from their predecessors, infer fundamentals (μ_1^*, μ_2^*) , then formulate their stopping rule as if they are playing the optimal-stopping problem without reference dependence, with $(X_1, X_2) \sim \Psi(\mu_1^*, \mu_2^*; \gamma)$. When the optimal-stopping problem satisfies Assumption 1, their subjectively optimal stopping rule is a cutoff rule, which they commit to before drawing X_1 .

I now characterize the pseudo-true fundamentals in the inference problem combining the gambler's fallacy and neglect of reference dependence. Suppose Ψ, Ψ° are two (possibly different) joint distributions for (X_1, X_2) . Write $\mathcal{H}^Y(\Psi; \Psi^\circ; c^Y)$ for the distribution of observations when objectively $(X_1, X_2) \sim \Psi$, predecessors hold the subjective model that $(X_1, X_2) \sim \Psi^\circ$ form encoded experiences Y_1, Y_2 based on their expectations, with Y_2 observed if and only if $Y_1 < c^W$. The pseudo-true fundamentals are the solutions to the minimization problem:

$$\min_{(\mu_1, \mu_2)} D_{KL}(\mathcal{H}^Y(\Psi^\bullet; \Psi(\mu_1^\circ, \mu_2^\circ; \gamma); (1 + \eta)c - \eta\mu_1^\circ) || \mathcal{H}(\Psi(\mu_1, \mu_2; \gamma); (1 + \eta)c - \eta\mu_1^\circ)).$$

To interpret, what the agent actually observes are a censored copy of encoded draws where (X_1, X_2) are objectively distributed according to Ψ^\bullet , then encoded into (Y_1, Y_2) based on expectations formed using the subjective model $\Psi(\mu_1^\circ, \mu_2^\circ; \gamma)$. However, the agent thinks she is observing censored copy of raw draws from some distribution $\Psi(\mu_1, \mu_2; \gamma)$ and estimates (μ_1, μ_2) . The pseudo-true parameters therefore depend on c , the censoring threshold for X_1 , and on μ_1°, μ_2° , the predecessors' beliefs about the fundamentals.

Proposition OA.14. *The pseudo-true fundamentals are $\mu_1^* = (1 + \eta)\mu_1^\bullet - \eta\mu_1^\circ$ and*

$$\mu_2^* = (1 + \eta)\mu_2^\bullet - \eta\mu_2^\circ + \gamma[(1 + \eta)(\mu_1^\bullet - \mathbb{E}[X_1|X_1 \leq c]) + \eta(\mu_1^\circ - \mathbb{E}[X_1|X_1 \leq c])].$$

As we may expect, when $\eta = 0$ the inference reduces to $\mu_1^* = \mu_1^\bullet$, $\mu_2^* = \mu_2^\bullet - \gamma(\mu_1^\bullet - \mathbb{E}[X_1|X_1 \leq c])$, which is the pseudo-true fundamentals given by Proposition 2 for gambler's

fallacy agents without misattribution of reference dependence. When $\gamma = 0$, the agent's inference no longer depends on censoring, as we have $\mu_1^* = (1 + \eta)\mu_1^\bullet - \eta\mu_1^\circ$ and $\mu_2^* = (1 + \eta)\mu_2^\bullet - \eta\mu_2^\circ$. This special case resembles [Bushong and Gagnon-Bartsch \(2018\)](#)'s results for learning under misattribution, as previous generation's beliefs influence the coding of draws as elating or disappointing, so the overall estimates reflect a combination of the true fundamentals and these previous expectations.

When $\gamma > 0$, a more severely censored dataset leads to a more pessimistic estimate of the second-period fundamental. So the direction of the censoring effect is preserved. More importantly, the marginal effect of decreasing the acceptance threshold c on the second-period inference now depends on η . This comes from an interaction between the gambler's fallacy and misattribution of reference dependence. In the baseline case where the agent faces a dataset of raw histories, the sample mean of the uncensored second period draws is μ_2^\bullet regardless of the censoring threshold. But with misattribution of reference dependence, predecessors are on average more disappointed after low values of first-period draws, as they expect a greater reversal. So in fact, $Y_2|Y_1 = y_1$ is stochastically increasing in y_1 . A more heavily censored dataset not only leads to a more pessimistic second-period inference due to the usual censoring effect, but also decreases the sample mean of the uncensored second-period observations. The magnitude of this second distortion depends on η .

Proof. I first derive the objective distributions of Y_1 and Y_2 when $(X_1, X_2) \sim \Psi^\bullet$ objectively but predecessors encode the draws using expectations from the subjective model $\Psi(\mu_1^\circ, \mu_2^\circ; \gamma)$.

Since $Y_1 = (1 + \eta)X_1 - \eta\mu_1^\circ$, we have

$$Y_1 \sim \mathcal{N}((1 + \eta)\mu_1^\bullet - \eta\mu_1^\circ, (1 + \eta)^2\sigma^2).$$

Next, when $Y_1 = y_1$, $X_1 = \frac{y_1 + \eta\mu_1^\circ}{1 + \eta}$. So,

$$\begin{aligned} Y_2|(Y_1 = y_1) &= X_2 + \eta \left(X_2 - (\mu_2^\circ - \gamma(\frac{y_1 + \eta\mu_1^\circ}{1 + \eta} - \mu_1^\circ)) \right) \\ &= X_2 + \eta \left(X_2 - (\mu_2^\circ - \gamma(\frac{y_1 - \mu_1^\circ}{1 + \eta})) \right) \\ &= (1 + \eta)X_2 - \eta\mu_2^\circ + \eta\gamma\frac{y_1 - \mu_1^\circ}{1 + \eta}. \end{aligned}$$

Since X_2 is objectively independent of X_1 and hence of Y_1 , we have

$$Y_2|(Y_1 = y_1) \sim \mathcal{N} \left((1 + \eta)\mu_2^\bullet - \eta\mu_2^\circ + \eta\gamma\frac{y_1 - \mu_1^\circ}{1 + \eta}, (1 + \eta)^2\sigma^2 \right).$$

Using the objective distributions of Y_1, Y_2 and the censoring threshold $Y_1 \geq (1 + \eta)c - \eta\mu_1^\circ$,

we may rewrite the KL divergence

$$D_{KL}(\mathcal{H}^Y(\Psi^\bullet; \Psi(\mu_1^\circ, \mu_2^\circ; \gamma); (1+\eta)c - \eta\mu_1^\circ) || \mathcal{H}(\Psi(\mu_1, \mu_2; \gamma); (1+\eta)c - \eta\mu_1^\circ))$$

as:

$$D_{KL}(\mathcal{N}(\mu_1, \sigma^2); \mathcal{N}((1+\eta)\mu_1^\bullet - \eta\mu_1^\circ, (1+\eta)^2\sigma^2)) \\ + \int_{-\infty}^{(1+\eta)c - \eta\mu_1^\circ} \phi(y_1; (1+\eta)\mu_1^\bullet - \eta\mu_1^\circ, (1+\eta)^2\sigma^2) \cdot \\ D_{KL}(\mathcal{N}(\mu_2 - \gamma(y_1 - \mu_1), \sigma^2); \mathcal{N}((1+\eta)\mu_2^\bullet - \eta\mu_2^\circ + \eta\gamma\frac{y_1 - \mu_1^\circ}{1+\eta}, (1+\eta)^2\sigma^2)) dy_1.$$

Simplifying using the formula for KL divergence between two Gaussian random variables,

$$\ln\left(\frac{\sigma^2}{(1+\eta)^2\sigma^2}\right) + \frac{(1+\eta)^2\sigma^2 + (\mu_1 - (1+\eta)\mu_1^\bullet + \eta\mu_1^\circ)^2}{2\sigma^2} - \frac{1}{2} \\ + \int_{-\infty}^{(1+\eta)c - \eta\mu_1^\circ} \phi(y_1; (1+\eta)\mu_1^\bullet - \eta\mu_1^\circ, (1+\eta)^2\sigma^2) \cdot \\ \left(\ln\frac{\sigma^2}{(1+\eta)^2\sigma^2}\right) + \frac{(1+\eta)^2\sigma^2 + (\mu_2 - \gamma(y_1 - \mu_1) - (1+\eta)\mu_2^\bullet + \eta\mu_2^\circ - \eta\gamma\frac{y_1 - \mu_1^\circ}{1+\eta})^2}{2\sigma^2} - \frac{1}{2} dy_1.$$

Dropping terms not dependent on μ_1, μ_2 and re-normalizing, the pseudo-true fundamentals must minimize:

$$\xi(\mu_1, \mu_2) = (\mu_1 - (1+\eta)\mu_1^\bullet + \eta\mu_1^\circ)^2 \\ + \int_{-\infty}^{(1+\eta)c - \eta\mu_1^\circ} \phi(y_1; (1+\eta)\mu_1^\bullet - \eta\mu_1^\circ, (1+\eta)^2\sigma^2) \cdot \\ (\mu_2 - \gamma(y_1 - \mu_1) - (1+\eta)\mu_2^\bullet + \eta\mu_2^\circ - \eta\gamma\frac{y_1 - \mu_1^\circ}{1+\eta})^2 dy_1.$$

Observe that at the optimum (μ_1^*, μ_2^*) , FOC implies $\frac{\partial \xi}{\partial \mu_1}(\mu_1^*, \mu_2^*) = \frac{\partial \xi}{\partial \mu_2}(\mu_1^*, \mu_2^*) = 0$. But we also have $\frac{\partial \xi}{\partial \mu_1}(\mu_1^*, \mu_2^*) = 2(\mu_1^* - (1+\eta)\mu_1^\bullet + \eta\mu_1^\circ) + \gamma\frac{\partial \xi}{\partial \mu_2}(\mu_1^*, \mu_2^*)$, which shows $\mu_1^* = (1+\eta)\mu_1^\bullet - \eta\mu_1^\circ$. Then, straightforward algebra shows

$$\mu_2^* = (1+\eta)\mu_2^\bullet - \eta\mu_2^\circ - \gamma[(1+\eta)(\mu_1^\bullet - \mathbb{E}[X_1|X_1 \leq c]) + \eta(\mu_1^\circ - \mathbb{E}[X_1|X_1 \leq c])],$$

as desired. □

In the learning steady state incorporating reference-dependence neglect, we must have $\mu_1^* = \mu_1^\circ$ and $\mu_2^* = \mu_2^\circ$, that is the previous generation's beliefs match the pseudo-true fundamentals of the current generation. Making this substitution in Proposition [OA.14](#), we immediately find:

Corollary A.4. *In steady state $(\mu_1^{\infty,\eta}, \mu_2^{\infty,\eta}, c^{\infty,\eta})$ of the learning environment with misattribution of reference dependence, $\mu_1^{\infty,\eta} = \mu_1^\bullet$, $\mu_2^{\infty,\eta} = \mu_2^\bullet - \gamma \left(\frac{1+2\eta}{1+\eta} \right) (\mu_1^\bullet - \mathbb{E}[X_1 | X_1 \leq c^{\infty,\eta}])$.*

In the steady state, agents hold correct beliefs about first-period fundamental but pessimistic beliefs about second-period fundamental. We find an interaction effect between gambler's fallacy and reference-dependence neglect. Indeed, if $\gamma = 0$, then in steady state agents hold correct beliefs about both fundamentals. And if agents only have the gambler's fallacy but do not encode draws based on reference dependence, then the inference of agents observing a dataset censored according to the cutoff threshold $X_1 \geq c^{\infty,\eta}$ would be more less distorted than the steady-state belief $\mu_2^{\infty,\eta}$.

Finally, I compare the steady states with and without neglect of reference dependence, fixing the optimal-stopping problem. I find that provided the usual regularities in the main text of the paper are satisfied, adding reference-dependence neglect leads to more pessimistic beliefs and lower stopping threshold in the steady state.

Proposition OA.15. *Consider any optimal-stopping problem satisfying Assumptions 1 and 2. Then there is a unique steady state, $(\mu_1^\infty, \mu_2^\infty, c^\infty)$. For any steady state $(\mu_1^{\infty,\eta}, \mu_2^{\infty,\eta}, c^{\infty,\eta})$ of the society with $\eta > 0$ facing the same optimal-stopping problem, we have $\mu_2^{\infty,\eta} < \mu_2^\infty$ and $c^{\infty,\eta} < c^\infty$.*

Proof. From Corollary A.4, $\mu_1^{\infty,\eta} = \mu_1^\infty = \mu_1^\bullet$.

By way of contradiction, suppose $\mu_2^{\infty,\eta} \geq \mu_2^\infty$. Consider a baseline society (with $\eta = 0$) whose generation 1 agents have the beliefs $\mu_{1,[1]}^* = \mu_1^\bullet$, $\mu_{2,[1]}^* = \mu_2^{\infty,\eta}$. In the next generation, we must have $\mu_{2,[2]}^* = \mu_2^*(C(\mu_1^\bullet, \mu_2^{\infty,\eta}; \gamma))$. By monotone convergence of beliefs across generations in the baseline model (Proposition 6), $\mu_{2,[2]}^* \leq \mu_2^*(C(\mu_1^\bullet, \mu_2^{\infty,\eta}; \gamma)) \leq \mu_2^{\infty,\eta}$. That is, $\mu_2^\bullet - \gamma(\mu_1^\bullet - \mathbb{E}[X_1 | X_1 \leq C(\mu_1^\bullet, \mu_2^{\infty,\eta}; \gamma)]) \leq \mu_2^{\infty,\eta}$.

Since $(\mu_1^{\infty,\eta}, \mu_2^{\infty,\eta}, c^{\infty,\eta})$ is a steady state for a society with $\eta > 0$, $c^{\infty,\eta} = C(\mu_1^\bullet, \mu_2^{\infty,\eta}; \gamma)$. Also, from Corollary A.4, we get $\mu_2^{\infty,\eta} = \mu_2^\bullet - \gamma \left(\frac{1+2\eta}{1+\eta} \right) (\mu_1^\bullet - \mathbb{E}[X_1 | X_1 \leq c^{\infty,\eta}])$. Since $\gamma > 0$, this implies $\mu_1^\bullet - \mathbb{E}[X_1 | X_1 \leq c^{\infty,\eta}] \geq \left(\frac{1+2\eta}{1+\eta} \right) (\mu_1^\bullet - \mathbb{E}[X_1 | X_1 \leq c^{\infty,\eta}])$. This is a contradiction since $\mu_1^\bullet - \mathbb{E}[X_1 | X_1 \leq c^{\infty,\eta}] > 0$ and $\left(\frac{1+2\eta}{1+\eta} \right) > 1$ since $\eta > 0$. \square

OA 3.4 Only Observing the Final Draw

In the baseline model, the history h_n of predecessor $n \in [0, 1]$ records just the first-period draw $h_n = (x_{1,n}, \emptyset)$ if n stopped in period 1, and it records both draws $h_n = (x_{1,n}, x_{2,n})$ if n continued into period 2. An *outcome history* differs from a history of the baseline model in that it always records only one draw – the one from the period where the agent stops. So, predecessor n 's outcome history h_n^o is either $h_n^o = (x_{1,n}, \emptyset)$ or $h_n^o = (\emptyset, x_{2,n})$. This kind of observation may be natural when the optimal-stopping problem is search without recall

(i.e. Example 1 with $q = 0$) and managers in the current generation only know about the candidates who were eventually hired in the previous generation across various firms, but not the early-phase candidates who were discovered but let go.

Write $\mathcal{H}^o(\Psi^\bullet; c)$ for the distribution of predecessors' outcome histories when $(X_1, X_2) \sim \Psi^\bullet$ and predecessors use the cutoff threshold c . I show that for agents using a method-of-moments (MOM) inference procedure analogous to the one in Appendix C, they will still infer the pseudo-true fundamentals associated with usual history distribution $\mathcal{H}(\Psi^\bullet; c)$ in the baseline model. To be precise, MOM agents find μ_1^{MO}, μ_2^{MO} so that $\mathcal{H}^o(\Psi(\mu_1^{MO}, \mu_2^{MO}; \gamma); c)$ matches $\mathcal{H}^o(\Psi^\bullet; c)$ in terms of the sample means of the uncensored first-period draws and uncensored second-period draws. As $\mu_1 \mapsto \mathbb{E}_{\tilde{X} \sim \mathcal{N}(\mu_1, \sigma)}[\tilde{X} | \tilde{X} \geq c]$ is a strictly increasing function, the MOM inference μ_1^{MO} must correctly estimate the first-period fundamental, $\mu_1^{MO} = \mu_1^\bullet$. Also, note that for any $\hat{\mu}_1, \hat{\mu}_2 \in \mathbb{R}$ and any $\hat{\gamma} \geq 0$, the second moments is the same in the outcome histories distribution $\mathcal{H}^o(\Psi(\hat{\mu}_1, \hat{\mu}_2; \hat{\gamma}); c)$ as in the baseline histories distribution $\mathcal{H}(\Psi(\hat{\mu}_1, \hat{\mu}_2; \hat{\gamma}); c)$. By the method-of-moments interpretation of $\mu_2^*(c)$, we conclude that $\mu_2^{MO}(c) = \mu_2^*(c)$ for all $c \in \mathbb{R}$.

The KL-divergence minimizing pseudo-fundamentals for agents observing outcomes proves difficult to calculate analytically. This is because the likelihood of the outcome history $h_n^o = (\emptyset, x_{2,n})$ is given by an integral over its likelihoods for different censored realizations of X_1 . Using numerical simulations, I show in Section OA 4.3 that when Bayesian agents with a correct dogmatic belief about μ_1^\bullet face a large, finite dataset of outcome histories, their inference about the second-period fundamental seems to closely match $\mu_2^*(c)$. It remains as an open conjecture whether the minimizers in these two different KL-divergence minimization problems in fact coincide exactly.

OA 4 Numerical Simulations

OA 4.1 Pessimism and Fictitious Variation in Finite Datasets

Lemma A.1 proves that when an agent with a full-support prior $g : \mathbb{R}^2 \rightarrow \mathbb{R}$ observes N histories drawn from the distribution $\mathcal{H}^\bullet(c)$, then as N goes to infinity her posterior belief almost surely concentrates on the KL-divergence minimizing pseudo-true parameters. In this section, I use simulations to check how well the predictions of Proposition 2 and Proposition 8 hold up in finite datasets. In particular, I am interested in the pessimistic inference in Proposition 2 and the fictitious variation in Proposition 8.

I consider the objective distribution $(X_1, X_2) \sim \Psi(\mu_1, \mu_2, \sigma^2, \sigma^2; 0)$ with $\mu_1 = \mu_2 = 0$, $\sigma^2 = 1$, and a stopping rule that censors X_2 whenever $X_1 \geq 1$. I suppose agents have dogmatic belief in $\gamma = 0.5$ and start with the (improper) flat prior on \mathbb{R}^2 . In Figures

OA.1 and OA.2, I plot distributions of the Bayesian posterior mode after a dataset of size $N = 100, 1000, 10000$. I find that when $N = 100$, there is 91.9% chance that agents underestimate the second-period and and 78.9% chance they believe in fictitious variation for the second-period conditional variance. These probabilities grow to virtually 100% for $N = 1000$ and $N = 10000$.

OA 4.2 Welfare Implications of Endogenous Learning

In this paper, I have emphasized the dynamics of mislearning and the interaction between distorted stopping strategy and distorted beliefs under the gambler’s fallacy. The positive feedback cycle between censoring and gambler’s fallacy leads to additional welfare implications beyond what would happen with gambler’s fallacy alone in a static, exogenous-data setting. Figure OA.3 returns to the illustrative example used for Figure 1 and compares the expected loss (relative to using the objectively optimal stopping rule) in the learning steady state versus the expected loss for the first-generation agents. Recall that Figure 1 considers search without recall, so $u_1(x_1) = x_1$, $u_2(x_1, x_2) = x_2$ with true fundamentals $\mu_1^\bullet = \mu_2^\bullet = 0$. As I initialize the 0th generation with the objectively optimal stopping threshold $c_{[0]} = 0$, misinformation from the gambler’s fallacy is solely responsible the first-generation loss. The long-run loss, on the other hand, is exacerbated by successive generations of predecessors lowering their stopping threshold and thus censoring the dataset with increasing severity. As Figure OA.3 shows, the fraction of long-run losses attributable to passive inference under gambler’s fallacy falls with the degree of the bias, highlighting the need of the dynamic analysis especially in environment where we expect the bias to be more serious.

OA 4.3 Inference of Misspecified Bayesian Agents when Observing Only the Final Draw

Consider a Bayesian agent with the (improper) flat prior over the class of models

$$\{\Psi(\mu_1, \mu_2, \sigma^2, \sigma^2, \gamma) : \mu_1 = 0, \sigma^2 = 1, \gamma = 0.5, \mu_2 \in \mathbb{R}\}.$$

Suppose she sees a dataset of 10,000 predecessor outcome histories – that is, histories that only show the draw for the period where the predecessor stopped, as in Section OA 3.4 – when predecessors use the stopping strategy S_c . Figure OA.4 presents the expected posterior mode²⁷ for different cutoff thresholds, $c \in \{-2, -1.5, -1, -0.5, 0, 0.5, 1, 1.5, 2\}$. These

²⁷These expected posterior modes were calculated by performing 1000 simulation at each cutoff threshold, computing the posterior mode in each simulation, then taking the average. The standard error for each of these empirical averages is smaller than 0.002.

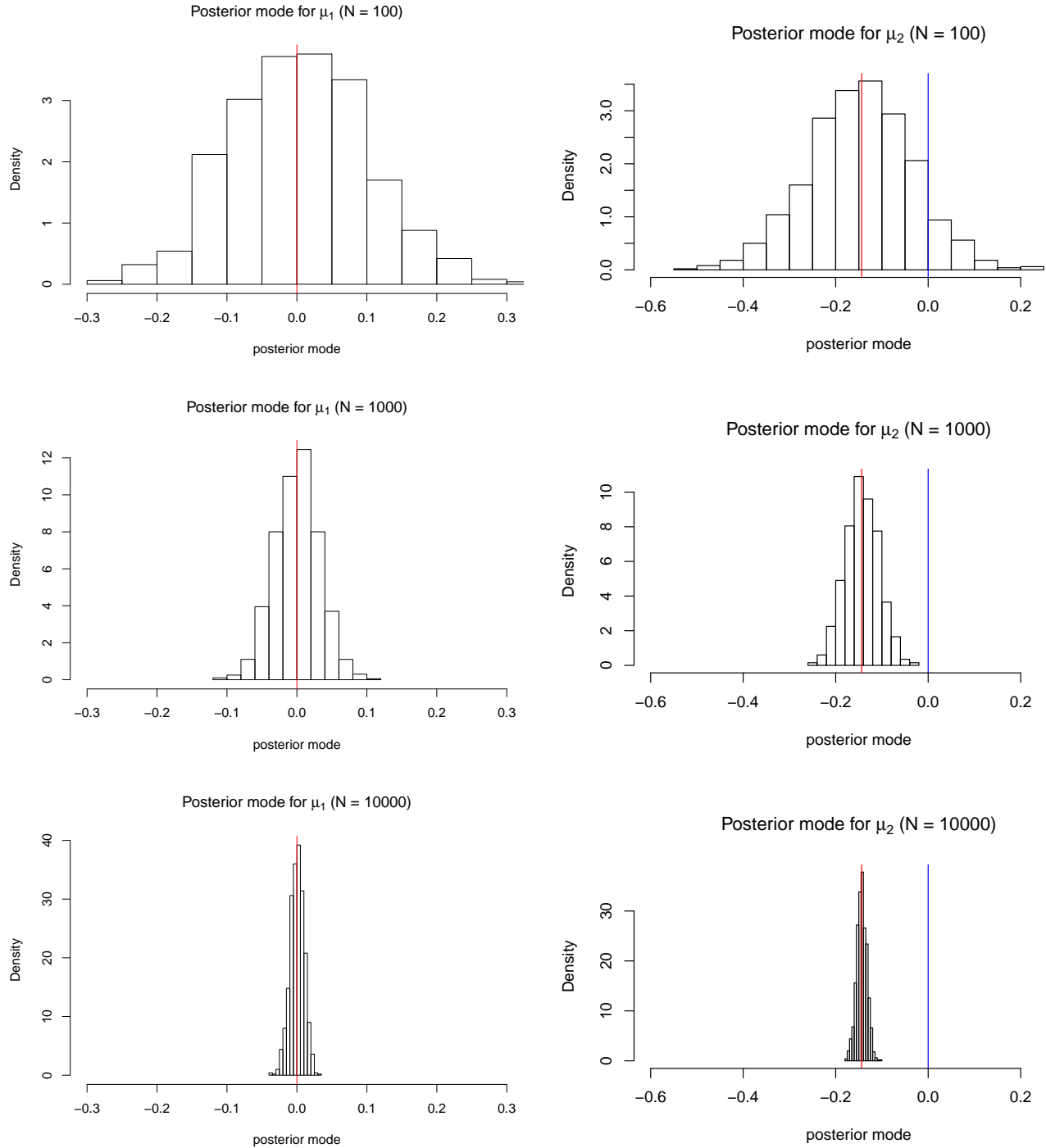


Figure OA.1: Histograms of inferences about fundamentals in finite datasets. The red lines in the histograms for μ_1 denote the pseudo-true fundamental (and also the true fundamental) $\mu_1^*(c = 1) = 0$. The blue lines in the histograms for μ_2 denote the true fundamental $\mu_2^* = 0$, while the red lines show the pseudo-true fundamental $\mu_2^*(c = 1) = -0.1438$.

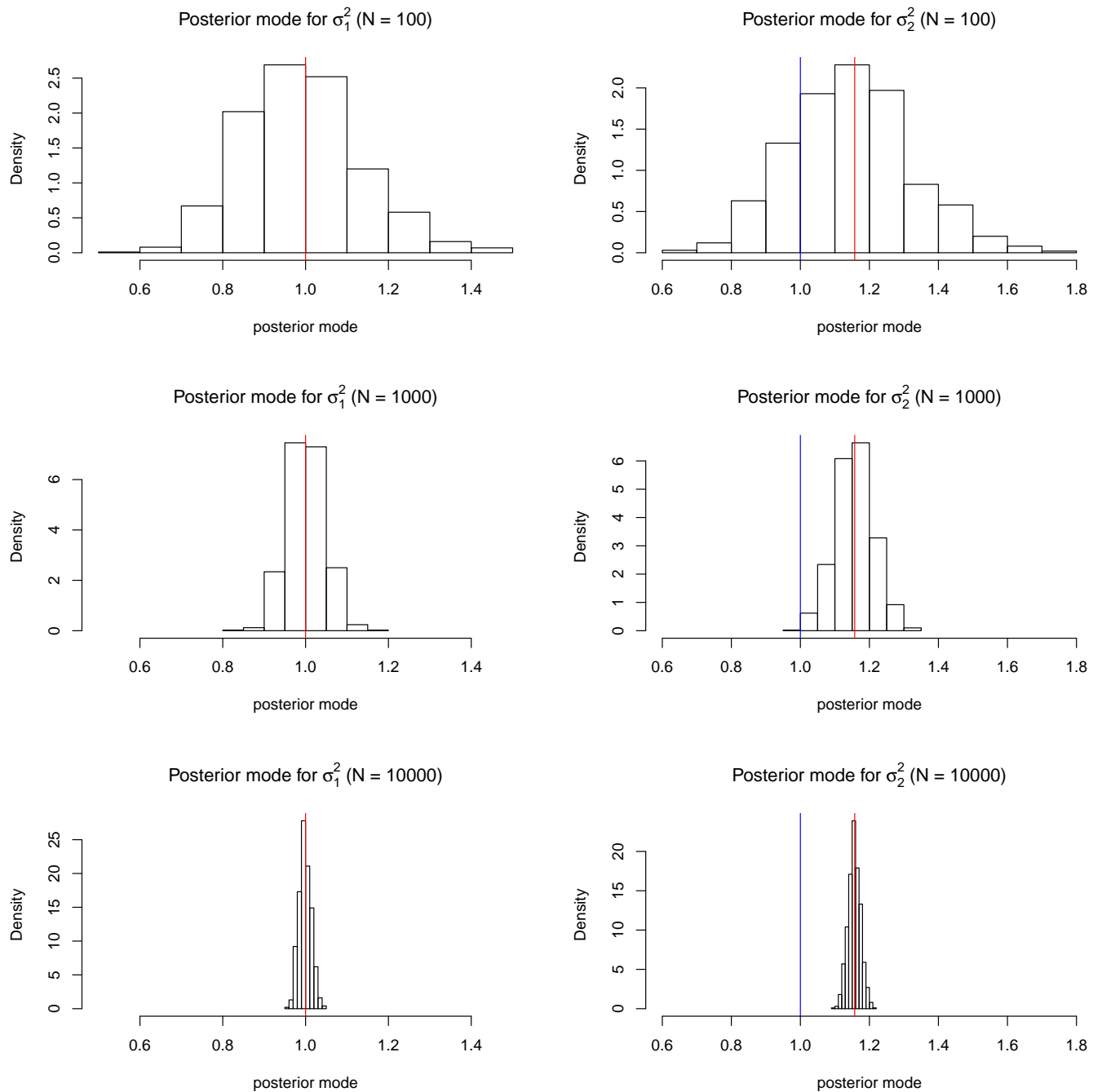


Figure OA.2: Histograms of inferences about variances in finite datasets. The red lines in the histograms for σ_1^2 denote the pseudo-true variance (and also the true variance) $(\sigma_1^*)^2(c = 1) = 1$. The blue lines in the histograms for σ_2^2 denote the true fundamental $(\sigma_2^\bullet)^2 = 1$, while the red lines show the pseudo-true fundamental $(\sigma_2^*)^2(c = 1) = 1.157$.

Positive feedback amplifies first-generation loss

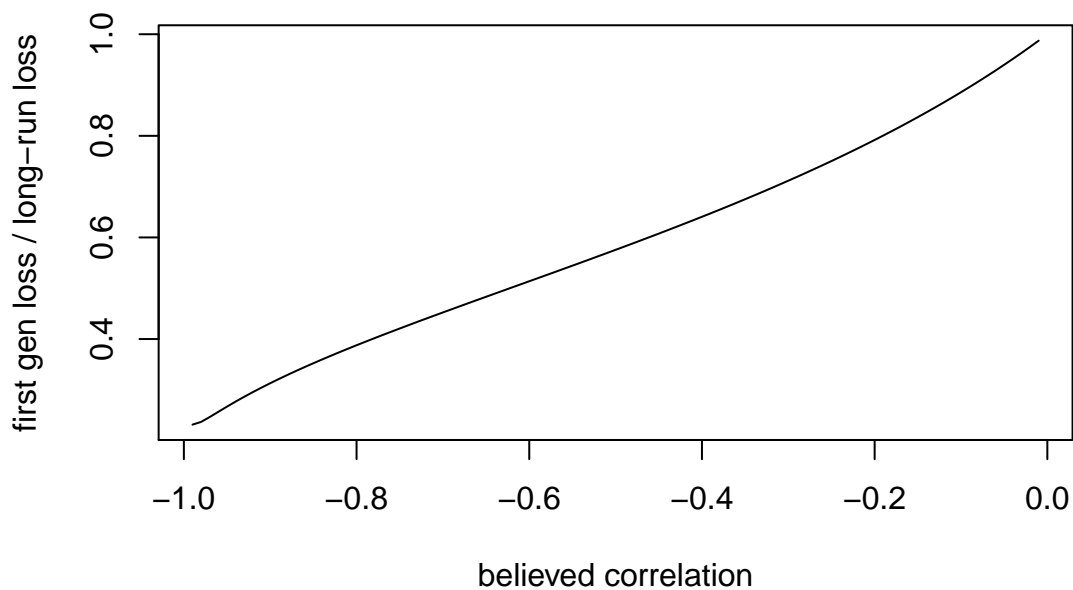


Figure OA.3: Welfare loss in the first generation relative to the long-run welfare loss, as a function of the believed correlation between X_1 and X_2 . A more negative correlation corresponds to a larger γ and a more severe gambler's fallacy bias.

Inference from Baseline Histories and Outcome Histories

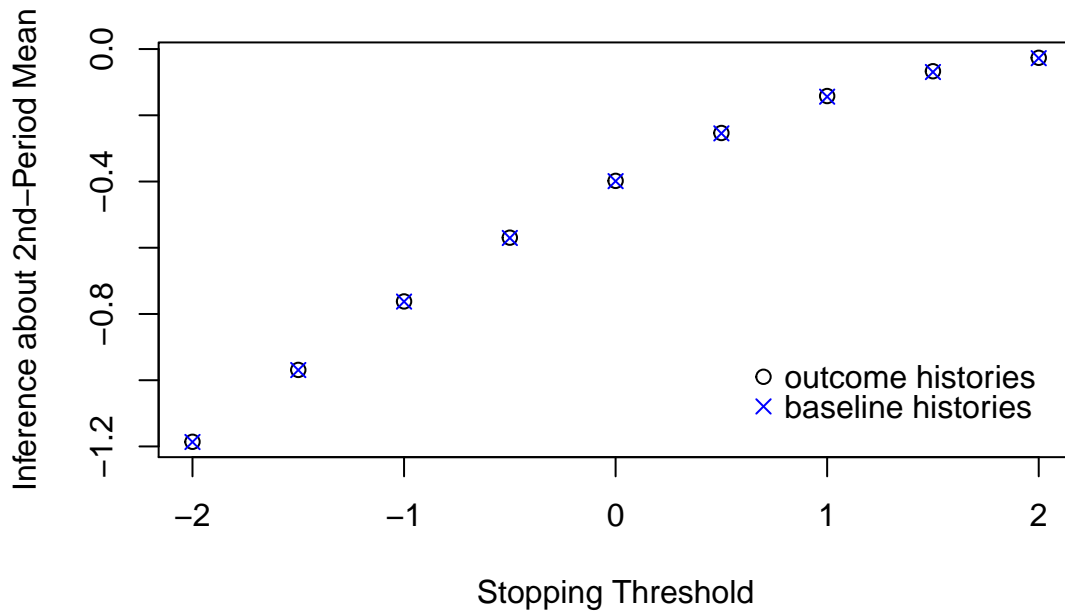


Figure OA.4: Black circles show the expected posterior modes for Bayesian agents with a flat prior over μ_2 after seeing 10,000 outcome histories generated according to different stopping thresholds. Blue X's show the pseudo-true second-period fundamentals $\mu_2^*(c)$ in the baseline model for different values of c .

numerical calculations give suggestive evidence that when agents hold correct beliefs about the first-period fundamental, they make more pessimistic inferences about the second-period fundamentals in outcome histories datasets with a lower stopping threshold. That is, the direction of the baseline censoring effect continues to obtain in this alternative observation structure. In fact, they also suggest the pseudo-true second-period fundamentals with outcome histories are very similar to the pseudo-true second-period fundamentals in the baseline model.