Higher-Order Beliefs and (Mis)learning from Prices*

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Abstract

We study misperceptions of private-signal correlation in a setting where a population repeatedly matches up to play an incomplete-information Cournot duopoly game. We show that a misperception's viability can depend on whether agents hold flexible or dogmatic beliefs about price elasticity. If agents have flexible beliefs and learn elasticity by observing prices, correlation misperceptions indirectly distort behavior through elasticity misinference. If agents dogmatically know elasticity with certainty, this learning channel is eliminated. The direct and indirect effects of correlation misperception on behavior oppose each other, implying that the possibility of elasticity inference can reverse an error's viability.

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1 Introduction

In strategic situations where players face uncertainty over the state of nature, agents' behavior can depend on both their beliefs over the state (i.e., first-order beliefs) and their beliefs over other players' information (i.e., higher-order beliefs). But, significant evidence suggests economic actors often find it difficult to form accurate higher-order beliefs or detect systematic biases in them. This paper investigates mistaken higher-order beliefs from an evolutionary perspective, asking which errors might confer an advantage and when.

Our main message is that whether a given misperception in higher-order beliefs over a payoff-relevant state improves or harms payoffs can depend on whether other, persistent parameters of the game are known or inferred. A higher-order misperception can directly affect the agent's action by influencing their conjecture of opponent behavior. But more subtly, when the agent does not know the persistent game parameters and must infer them through repeated play, the same mispredictions about opponents' behavior cause the agent to misinterpret game outcomes. This misinference induces distorted beliefs about the game parameters, possibly letting the agent commit to strategically beneficial behavior. So, in addition to its direct effect, higher-order misperceptions can distort behavior indirectly through this *learning channel*. Given that errors can have opposite direct and indirect effects on behavior, an agent's knowledge about the persistent parameters (and thus whether the learning channel is present) can determine whether the error facilitates such beneficial commitments.

We illustrate this idea in the context of a linear-quadratic-normal (LQN) Cournot duopoly game of incomplete information, similar to Vives (1988).[1] The state is the intercept of the demand curve (i.e., demand shock), drawn i.i.d. from a normal distribution each time the game is played, with players receiving possibly correlated signals before choosing actions. The key persistent game parameter is the slope of the demand curve (i.e., price elasticity), which players may or may not know. Setups within the LQN family have received significant attention in part because they admit tractable comparative statics with respect to players' information (illustrated in Bergemann and Morris (2013), as well as Miyashita and Ui (2023); Bergemann, Heumann, and Morris (2017)). Here, we use this setup to study misperceptions regarding the signals others observe.

[1]This model is extended to more general structures by Angeletos and Pavan (2007). We explain how our results generalize in the Supplemental Appendix.

To formalize the consequences of higher-order misperceptions, we apply stability concepts from the literature on the *indirect evolutionary approach* (surveyed in Alger and Weibull (2019); Robson and Samuelson (2011)) to such errors. Typically, this approach assumes individuals are endowed with different subjective preferences over game outcomes. If a "new preference" leads to higher objective payoffs in equilibrium than an "existing preference" when the latter is dominant in the society, then we say the former has an evolutionary advantage and can invade the latter. In our application, a higher-order misperception is equivalent to a subjective preference only if the agent knows the persistent game parameters. If the agent is uncertain about these parameters and infers them from outcomes, then the same misperception can lead to different beliefs about these parameters when the environment varies, as highlighted in our companion paper He and Libgober (2024b). A recent literature on misspecified Bayesian learning (Esponda and Pouzo (2016); Frick, Iijima, and Ishii (2024); Heidhues, Koszegi, and Strack (2018), among others) studies the implications of mislearning persistent parameters of the environment on behavior and welfare. Our contribution is to explore how uncertainty about such persistent parameters (that is, the possibility of mislearning) affects the equilibrium consequences of misperceiving others' information within a seminal game.

We consider a society with incumbents who know the true correlation between players' signals about the i.i.d. demand shocks. If agents are certain about the persistent price elasticity, then assortative matching (i.e., entrants' welfare is determined by how they do when playing against each other) favors entrants who overestimate correlation in different players' signals, while uniform matching (i.e., entrants' welfare is determined by how they do when playing against incumbents) favors entrants who underestimate said correlation. However, the situation is exactly reversed when agents do not know price elasticity and learn from game outcomes. That is, sometimes erroneous higher-order beliefs only benefit agents who are uncertain about the game's persistent parameters.

To see the intuition behind this result, consider a duopolist who misperceives signals to be more correlated than the truth—an error we refer to as *projection bias*.[2] It turns out the welfare implications of projection bias with uniform matching depend on whether the bias induces more aggressive strategies in equilibrium — that is, strategies that respond more to changes in private information. Thus, we examine whether projection bias increases the

[2] Other papers in economic theory studying the implications of projection bias include Gagnon-Bartsch, Pagnozzi, and Rosato (2021); Gagnon-Bartsch and Rosato (2024); Madarász (2012).

aggressiveness of subjective best responses.

On the one hand, the direct effect of projection bias makes the duopolist act less aggressively. When he has a private signal that suggests high market demand, he overestimates the similarity of the opponent's information and thus exaggerates how much the other player will increase their production level. This force limits how much the duopolist wishes to increase production, since the two firms' actions are strategic substitutes. The bias thus harms the duopolist's profits when he knows price elasticity.

On the other hand, the indirect effect of projection bias through the learning channel acts in the opposite direction. Suppose this duopolist infers elasticity from prices. Then, critically, projection bias causes the duopolist to underestimate price elasticity. This is because the duopolist's bias leads him to overestimate how steeply the opponent's production quantity changes as a function of the duopolist's signal realization. After a high private signal, the market price remains higher than the duopolist expects, which he rationalizes by inferring a low price elasticity. Underinferring price elasticity increases the aggressiveness of the duopolist's best response, as he underestimates how quickly the price decreases when he produces more.

We show that the indirect effect is stronger than the direct effect — intuitively since elasticity influences strategies much more than perceived signal correlation. However, the indirect effect is present only when agents are initially uncertain about price elasticity. Putting everything together, we conclude that projection bias can only invade a rational society when the entrants draw inferences about price elasticity from outcomes, not when they already know price elasticity with certainty.

2 Framework

Following the indirect evolutionary approach and our companion paper He and Libgober (2024b), we study an environment where a continuum of agents are matched up in pairs each period to play a two-player *stage game*.

2.1 Stage Game and Information Structure

We first describe the stage game. There is a demand state $\omega \sim \mathcal{N}(0, \sigma_\omega^2)$, where $\mathcal{N}(\mu, \sigma^2)$ is the normal distribution with mean μ and variance σ^2. Firm i receives a private signal $s_i = \omega + \epsilon_i$, and then chooses a quantity $q_i \in \mathbb{R}$. The resulting market price is $P = \omega - r^\bullet \cdot \frac{1}{2}(q_1 + q_2) + \zeta$, where $\zeta \sim \mathcal{N}(0, (\sigma_\zeta^\bullet)^2)$ is an idiosyncratic independent price shock. Firm i's profit is $q_i P - \frac{1}{2} q_i^2$.

As in many other LQN oligopoly models, market prices and quantity choices may be positive or negative. To interpret, when $P > 0$, the market pays for each unit of good supplied, and the market price decreases in total supply. When $P < 0$, the market pays for disposal. The cost $\frac{1}{2} q_i^2$ represents either a convex production cost or a convex disposal cost, depending on the sign of q_i.

We allow signals to be correlated conditional on ω and study the perception of this correlation. Recalling that $s_i = \omega + \epsilon_i$, we assume in particular that:

$$\epsilon_i = \frac{\kappa^\bullet}{\sqrt{(\kappa^\bullet)^2 + (1-\kappa^\bullet)^2}} z + \frac{1-\kappa^\bullet}{\sqrt{(\kappa^\bullet)^2 + (1-\kappa^\bullet)^2}} \eta_i,$$

where $\eta_i \sim \mathcal{N}(0, \sigma_\epsilon^2)$ is the idiosyncratic component generated i.i.d. across players and $z \sim \mathcal{N}(0, \sigma_\epsilon^2)$ is the common component. Higher $\kappa^\bullet$ leads to an information structure with higher conditional correlation. When $\kappa^\bullet = 0$, s_i and s_{-i} are conditionally uncorrelated given ω. When $\kappa^\bullet = 1$, we always have $s_i = s_{-i}$. This functional form for ϵ_i ensures $\text{Var}(s_i)$ is constant in $\kappa^\bullet$.

The persistent parameters of the stage game are $\sigma_\omega^2 > 0$ (variance of demand state), $r^\bullet > 0$ (a measure of the elasticity of market price with respect to quantity supplied), $(\sigma_\zeta^\bullet)^2 > 0$ (variance of price shock), and $\kappa^\bullet \in [0, 1]$ (a measure of signal correlation). Each time the stage game is played, ω, z, η_i and ζ are independently drawn from their respective distributions.

2.2 Models, Inference, and Strategies

The stage game is common knowledge except for parameters $\kappa^\bullet, r^\bullet$, and $(\sigma_\zeta^\bullet)^2$. Agents interpret their environment through their *models* of the world. A model can have two kinds of parameters: *free parameters* are estimated using game outcomes, while *fixed parameters* are dogmatically given by the model and not subject to inference. Signal correlation is a fixed parameter in every model, so different models can encode different dogmatic beliefs about that aspect of the stage game. We consider both *flexible* models where signal correlation $\tilde{\kappa}$ is a fixed parameter but price elasticity $\tilde{r}$ and price shock variance $\tilde{\sigma}_\zeta^2$ are free parameters,[3] as

[3]While a flexible model allows agents to infer both r and σ_ζ^2, their misinference about r drives the results. Since each player's profit is linear in the market price, belief about the variance of the idiosyncratic price shock does not change their expected payoffs or behavior. The parameter σ_ζ^2 absorbs changes in the variance

well as *dogmatic* models where $\tilde{\kappa}, \tilde{r}, \tilde{\sigma}_\zeta^2$ are all fixed parameters.

Our interest will be in studying misperceptions of signal correlation:

Definition 1. Let $\tilde{\kappa}$ be a player's perceived κ. A player exhibits *correlation neglect* if $\tilde{\kappa} < \kappa^\bullet$. A player exhibits *projection bias* if $\tilde{\kappa} > \kappa^\bullet$.

Correlation neglect agents underestimate the correlation in players' signals in the stage game, whereas projection bias agents exaggerate the similarity between others' information and their own, overestimating signal correlation. We are agnostic about the origin of these misspecifications. Instead, we ask whether such misspecifications can invade a rational society.

We now describe inference for flexible models. A *consequence* is a triple (s_i, q_i, P) that contains i's signal, i's quantity choice, and the realized market price. A *strategy* for i is a quantity choice as a function of i's signal realization, $Q_i(s_i)$. Let $\mathbb{Y}$ denote the set of all consequences, and let $\mathbb{S}$ denote the space of strategies. For each $(\kappa, r, \sigma_\zeta^2)$, we define $F_{\kappa,r,\sigma_\zeta^2} : \mathbb{S} \times \mathbb{S} \to \Delta(\mathbb{Y})$ to be the mapping between strategy profiles and the distribution over i's consequences in a stage game with parameters $(\kappa, r, \sigma_\zeta^2)$. We consider the following notion of free-parameter estimation for a given stage-game strategy profile:

Definition 2. Let $F^\bullet(Q_i, Q_{-i})$ denote the objective distribution over i's consequences given strategy profile Q_i, Q_{-i}. We say that inference $(\tilde{r}, \tilde{\sigma}_\zeta^2)$ is a *self-confirming inference given strategy profile* Q_i, Q_{-i} *and correlation* κ if $F^\bullet(Q_i, Q_{-i}) = F_{\kappa,\tilde{r},\tilde{\sigma}_\zeta^2}(Q_i, Q_{-i})$.

Self-confirming inferences are not falsified by the distribution of consequences that a player sees whenever they perceive correlation κ and repeatedly play the stage game using strategy Q_i against different opponents who all use the strategy Q_{-i}. Self-confirming inferences need not exist in general, in which case a goodness-of-fit criterion is needed for inferences to be well-defined. Esponda and Pouzo (2016) motivate KL-divergence as a natural criterion for misspecified Bayesian agents. But to avoid complications, we focus on true parameter values such that self-confirming inferences exist.

Next, we present a partial equilibrium notion where both players choose strategies that maximize utility given their beliefs about the persistent parameters, and said beliefs for some player i are are either the fixed parameters $\tilde{\kappa}, \tilde{r}, \tilde{\sigma}_\zeta^2$ (if i has a dogmatic model) or fixed parameter $\tilde{\kappa}$ together with the self-confirming inferences (if i has a flexible model). The reason why we do not require $-i$ to also derive beliefs from the same interaction is that there

of market price, creating significant tractability.

are some environments (i.e., when i is part of a negligible subpopulation) where $-i$'s beliefs are primarily shaped by the consequences they observe in other kinds of matches.

Definition 3. A strategy profile Q_i, Q_{-i} and belief profile $(\tilde{\kappa}_i, \tilde{r}_i, \tilde{\sigma}^2_{\zeta,i})$, $(\tilde{\kappa}_{-i}, \tilde{r}_{-i}, \tilde{\sigma}^2_{\zeta,-i})$ are a *linear partial equilibrium* if

- For each player k, $Q_k(s_k) = \alpha_k s_k$ for some $\alpha_k \geq 0$.
- For each player k, Q_k is an interim-stage best response against the opponent's strategy given belief $(\tilde{\kappa}_k, \tilde{r}_k, \tilde{\sigma}^2_{\zeta,k})$.
- For the first player i, $\tilde{\kappa}_i$ is the fixed parameter given by i's model, and $(\tilde{r}_i, \tilde{\sigma}^2_{\zeta,i})$ are either the fixed parameters given by i's dogmatic model or i's self-confirming inference given Q_i, Q_{-i}, and $\tilde{\kappa}_i$ (when i has a flexible model).

Since the best response (among the family of all strategies) to any linear strategy is linear for any belief about the correlation parameter and market price elasticity (shown in Lemma 2), we focus on equilibria where everyone uses linear strategies. We sometimes refer to the linear strategy $s_i \mapsto \alpha_i s_i$ simply as α_i.

2.3 Stability and Invasion

Our analysis will compare the *entrant model*, which is used by an infinitesimally small group of *entrants* in the population, with the *resident model*, which is used by the remaining group called the *residents*. We will consider two kinds of interaction structures. In uniform matching, each agent is matched with an opponent drawn uniformly at random from the entire population (and agents can observe their opponent's model). So, agents are only matched against the entrant group with infinitesimal probability. In assortative matching, agents are always matched within the group that uses the same model.[4]

In what follows, we use the subscript R to refer to the resident and the subscript E to refer to the entrant. For example, κ_R denotes the resident's perceived correlation parameter, and κ_E denotes that of the entrant. We let $\alpha_{g \to g'}$ denote the strategy that a group g agent uses when matched against someone from group g'. For strategies α_g, α_{-g} in the stage game,

[4]To avoid technical complications, we assume the entrants form an infinitesimally small group, as in He and Libgober (2024b). Our previous working paper (He and Libgober, 2024a) uses an alternative approach where the entrants forms a very small but positive-mass group, which does not change the results.

let $U^\bullet(\alpha_g, \alpha_{-g})$ be the objective expected utility of playing strategy α_g against α_{-g}. We refer to the objective expected utility of agents who use a model as that model's *fitness*.

Definition 4. With uniform matching, a *linear equilibrium* consists of strategies $\alpha_{R\to R}, \alpha_{R\to E}, \alpha_{E\to R}$ and beliefs $(\tilde{\kappa}_R, \tilde{r}_R, \tilde{\sigma}^2_{\zeta,R})$, $(\tilde{\kappa}_E, \tilde{r}_E, \tilde{\sigma}^2_{\zeta,E})$ such that:

- $\alpha_{R\to R}, \alpha_{R\to R}, (\tilde{\kappa}_R, \tilde{r}_R, \tilde{\sigma}^2_{\zeta,R}), (\tilde{\kappa}_R, \tilde{r}_R, \tilde{\sigma}^2_{\zeta,R})$ are a linear partial equilibrium,
- $\alpha_{E\to R}, \alpha_{R\to E}, (\tilde{\kappa}_E, \tilde{r}_E, \tilde{\sigma}^2_{\zeta,E}), (\tilde{\kappa}_R, \tilde{r}_R, \tilde{\sigma}^2_{\zeta,R})$ are a linear partial equilibrium.

We say κ_R is *resistant to invasion from κ_E with uniform matching* if $U^\bullet(\alpha_{R\to R}, \alpha_{R\to R}) \geq U^\bullet(\alpha_{E\to R}, \alpha_{R\to E})$ in every linear equilibrium, and we say κ_R is *susceptible to invasion with uniform matching* if $U^\bullet(\alpha_{R\to R}, \alpha_{R\to R}) < U^\bullet(\alpha_{E\to R}, \alpha_{R\to E})$ in every linear equilibrium.

With assortative matching, a *linear equilibrium* consists of strategies $\alpha_{R\to R}, \alpha_{E\to E}$ and beliefs $(\tilde{\kappa}_R, \tilde{r}_R, \tilde{\sigma}^2_{\zeta,R})$, $(\tilde{\kappa}_E, \tilde{r}_E, \tilde{\sigma}^2_{\zeta,E})$ such that:

- $\alpha_{R\to R}, \alpha_{R\to R}, (\tilde{\kappa}_R, \tilde{r}_R, \tilde{\sigma}^2_{\zeta,R}), (\tilde{\kappa}_R, \tilde{r}_R, \tilde{\sigma}^2_{\zeta,R})$ are a linear partial equilibrium,
- $\alpha_{E\to E}, \alpha_{E\to E}, (\tilde{\kappa}_E, \tilde{r}_E, \tilde{\sigma}^2_{\zeta,E}), (\tilde{\kappa}_E, \tilde{r}_E, \tilde{\sigma}^2_{\zeta,E})$ are a linear partial equilibrium.

We say κ_R is *resistant to invasion from κ_E with assortative matching* if $U^\bullet(\alpha_{R\to R}, \alpha_{R\to R}) \geq U^\bullet(\alpha_{E\to E}, \alpha_{E\to E})$ in every linear equilibrium, and we say κ_R is *susceptible to invasion with assortative matching* if $U^\bullet(\alpha_{R\to R}, \alpha_{R\to R}) < U^\bullet(\alpha_{E\to E}, \alpha_{E\to E})$ in every linear equilibrium.

This definition embeds the idea that agents with flexible models correctly think that the values of the persistent game parameters do not change depending on the group membership of the opponent. In particular, in an environment with uniform matching and with residents who have flexible models, the residents' beliefs about the free parameters $(\tilde{r}_R, \tilde{\sigma}^2_{\zeta,R})$ when playing against entrants are estimated using the consequences in their matches against other residents. This feature arises because these residents use all their available data to estimate the free parameters, and matches against entrants comprise an infinitesimally small portion of their data.

3 Subjective Best Responses and Self-Confirming Inferences

This section presents results characterizing best responses and self-confirming inferences. We first show that when i sees private signal s_i, their mean posterior beliefs about the state and about opponent's signal are linear functions of s_i.

Lemma 1. *There exists a strictly increasing function $\psi(\kappa)$, with $\psi(0) > 0$ and $\psi(1) = 1$, so that $\mathbb{E}_\kappa[s_{-i} \mid s_i] = \psi(\kappa) \cdot s_i$ for all $s_i \in \mathbb{R}$, $\kappa \in [0,1]$. Also, there exists a strictly positive $\gamma > 0$ so that $\mathbb{E}_\kappa[\omega \mid s_i] = \gamma \cdot s_i$ for all $s_i \in \mathbb{R}$, $\kappa \in [0,1]$.*

This result uses the tractability of the LQN framework. The coefficient γ that characterizes an agent's inference about the state does not depend on their perception of κ. But higher κ implies the agent infers more about the opponent's signal from their signal. In other words, a misperception of κ only distorts the agent's higher-order belief about the opponent's signal realization (and hence, opponent's belief), but does not affect the agent's first-order belief about the state ω. Linearity of $\mathbb{E}[\omega \mid s_i]$ and $\mathbb{E}[s_{-i} \mid s_i]$ in s_i gives us an explicit characterization of best responses in the stage game, given beliefs about the κ and r parameters:

Lemma 2. *For α_{-i} a linear strategy, i's expected utility from the linear strategy α_i given parameters $\kappa, r, \sigma_\zeta^2$ is $U_i(\alpha_i, \alpha_{-i}; \kappa, r) = \mathbb{E}[s_i^2] \cdot \left(\alpha_i\gamma - \frac{1}{2}r\alpha_i^2 - \frac{1}{2}r\psi(\kappa)\alpha_i\alpha_{-i} - \frac{1}{2}\alpha_i^2\right)$. For the same parameters, the linear strategy $\alpha_i^{BR}(\alpha_{-i}; \kappa, r) := \frac{\gamma - \frac{1}{2}r\psi(\kappa)\alpha_{-i}}{1+r}$ best responds to α_{-i} at the interim stage among all (possibly non-linear) strategies $Q_i : \mathbb{R} \to \mathbb{R}$.*

The key insight of Lemma 2 is that an agent's subjective expected utility and subjective best response depend on their beliefs about κ and r, but not σ_ζ^2. Call a linear strategy more *aggressive* if its coefficient $\alpha_i \geq 0$ is larger. Lemma 2 says agent i's subjective best response function becomes more aggressive when i believes in lower κ or lower r. The intuition for this was outlined in the introduction. We have $\frac{\partial \alpha_i^{BR}}{\partial \kappa} < 0$ as the agent can better leverage her private information about market demand when her rival does not share the same information. We have $\frac{\partial \alpha_i^{BR}}{\partial r} < 0$ because inelastic demand induces the agent to behave more aggressively, since prices become less responsive to quantity choices.

Finally, we characterize self-confirming inference given a strategy profile and a correlation perception.

Lemma 3. *There exists some $L > 0$ such that a unique self-confirming inference exists for any $\kappa \in [0,1]$ and $0 \leq \alpha_i, \alpha_{-i} \leq \gamma$ whenever $(\sigma_\zeta^\bullet)^2 \geq L$. The self-confirming inference for elasticity is $r_i^{INF}(\alpha_i, \alpha_{-i}, ; \kappa^\bullet, \kappa, r^\bullet) := r^\bullet \frac{\alpha_i + \alpha_{-i}\psi(\kappa^\bullet)}{\alpha_i + \alpha_{-i}\psi(\kappa)}$.*

Lemma 3 shows that for agents with flexible models, there is a unique inference of the free parameters r, σ_ζ^2 that perfectly matches the observed price distribution for any linear strategy profile, provided the true price shock variance is large enough and both agents' strategies are less aggressive than γ. Note that by Lemma 2, i's best response against any α_i is always bounded by γ, given any beliefs $\kappa \in [0,1], r \geq 0$. Therefore, no linear equilibrium exists where either player uses a strategy $\alpha_i > \gamma$ and we do not need to worry about excessively aggressive strategies. The self-confirming property always holds in the linear equilibria we use to define resistance to invasion.

A key lesson of Lemma 3 is that for a fixed strategy profile, misperceiving a higher signal correlation in the stage game causes the agent to infer a lower price elasticity, as suggested by the intuition in the introduction. This intuition will drive the interaction between misspecifications and inference in our main results in the next section.

4 Selecting Biases and the Role of the Learning Channel

We now turn to the selection of correlation perceptions and ask how the answer depends on whether agents have flexible models or dogmatic models. Throughout, we assume the true price shock variance exceeds the threshold L from Lemma 3. We first consider uniform matching with flexible models.

Proposition 1 (Uniform Matching Selects Projection Bias)**.** *Fix $r^\bullet > 0$, $\kappa^\bullet \in [0,1]$. For $(\sigma_\zeta^\bullet)^2 \geq L$, there exist $\underline{\kappa} < \kappa^\bullet < \bar{\kappa}$ so that taking $(\kappa_R, \kappa_E) = (\kappa^\bullet, \kappa)$ for any $\kappa \in [\underline{\kappa}, \bar{\kappa}]$ and assuming all agents have flexible models, there is a unique linear equilibrium with uniform matching. Furthermore, $\kappa^\bullet$ is susceptible to invasion with uniform matching if $\kappa > \kappa^\bullet$ and resistant to invasion with uniform matching if $\kappa < \kappa^\bullet$.*

The intuition for this result follows from the observation that projection bias generates a commitment to aggression as it leads the biased agents to under-infer market price elasticity. It is well-known that in Cournot oligopoly games, such commitment can be beneficial (Fershtman and Judd, 1987). Here, misspecification about signal correlation leads to misinference about elasticity, which causes the entrants to respond credibly to their opponents' play in an overly aggressive manner. The rational residents back down and yield a larger share of the surplus. However, projection bias is beneficial only in small measure, intuitively since excessive aggression can lead to overproduction past the point where such commitments are beneficial. Figure 1a illustrates this non-monotonicity of fitness in κ_E; while small increases

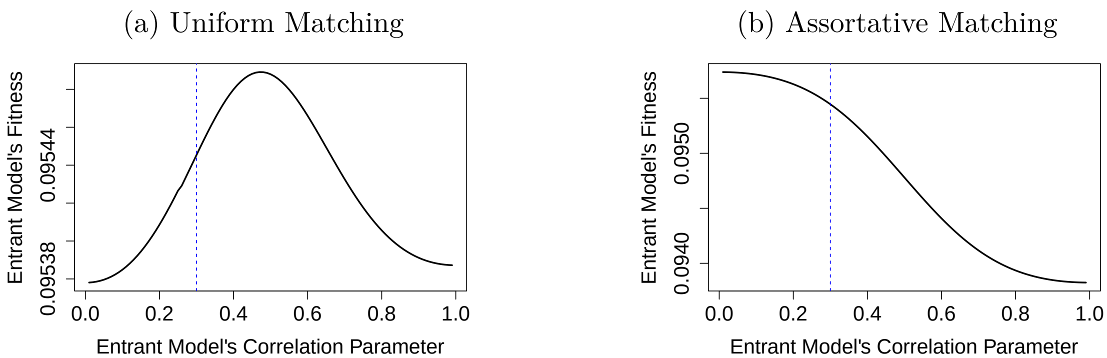

Figure 1: Fitness of the flexible entrant model against the correctly specified flexible resident model, as a function of the entrant model's perception of κ. The true parameters are $\kappa^\bullet = 0.3$, $r^\bullet = 1$, $\sigma_\omega^2 = \sigma_\epsilon^2 = 1$. The dashed vertical line marks the true correlation parameter.

in κ above $\kappa^\bullet$ increases entrant fitness, entrants no longer outperform residents with $\kappa = \kappa^\bullet$ once κ is close to one.

By contrast, assortative matching favors biases that lead to more *cooperative* behavior, and thus the commitment to aggression is detrimental to fitness. Correspondingly, we obtain the opposite result.

Proposition 2 (Perfectly Assortative Matching Selects Correlation Neglect)**.** *Fix* $r^\bullet > 0$ *and* $(\sigma_\zeta^\bullet)^2 \geq L$. *Assume all agents have flexible models. Then,* κ_R *is susceptible to invasion with assortative matching if* $\kappa_E < \kappa_R$*, and it is resistant to invasion with assortative matching if* $\kappa_E > \kappa_R$.

Correlation neglect leads agents with flexible models to over-infer elasticity, enabling commitment to less aggressive behavior. The contrast with uniform matching is illustrated in Figure 1b, where the entrant's fitness is not only locally decreasing in κ around $\kappa^\bullet$ but monotonic decreasing for all κ. Let α^{TEAM} denote the symmetric linear strategy profile that maximizes the sum of the two firms' expected objective payoffs. The proof of Proposition 2 shows that among symmetric strategy profiles, players' payoffs strictly decrease in aggressiveness in the region $\alpha > \alpha^{TEAM}$. For assortative matching and any $\kappa \in [0, 1]$, the linear equilibrium behavior in a group with correlation perception κ strictly increases in aggressiveness as κ grows, and this equilibrium play is always strictly more aggressive than α^{TEAM}. Lowering the perception of κ *always* confers an evolutionary advantage by bringing equilibrium play closer to α^{TEAM}.

So far, we have discussed the selection of correlation misperceptions when agents have flexible models. As mentioned before, the direct effect and the indirect effect of correlation misperception go in opposite directions. The next result shows that if agents have dogmatic models and the learning channel is shut down, then the conclusions of Propositions 1 and 2 can be reversed:

Proposition 3. *Let* $\kappa^\bullet \in [0,1]$*,* $r^\bullet > 0$*,* $(\sigma_\zeta^\bullet)^2 \geq L$ *be given and suppose all agents have dogmatic models whose fixed parameters about price elasticity and price shock variance are correct:* $(\tilde{r}, \tilde{\sigma}_\zeta^2) = (r^\bullet, (\sigma_\zeta^\bullet)^2)$*. Suppose* $\kappa_R = \kappa^\bullet$*. There exists* $\epsilon > 0$ *so that for any* $\kappa_l, \kappa_h \in [0,1]$*,* $\kappa_l < \kappa^\bullet < \kappa_h \leq \kappa^\bullet + \epsilon$*,* κ_R *is resistant to invasion from entrants with* $\kappa_E = \kappa_h$ *under uniform matching, and resistant to invasion from entrants with* $\kappa_E = \kappa_l$ *under assortative matching.*

Proposition 3 shuts down the learning channel using the assumption of dogmatic and correct beliefs about r. It implies that in the environments studied in Propositions 1 and 2, the beneficial misperceptions of κ confer their evolutionary advantage through the indirect effect of elasticity misinference. This force is stronger than the direct effect of the κ misperception, but it is only present when agents use flexible models. Thus, whether an error in higher-order beliefs can persist in a rational society may depend on whether the biased agents are open-minded or dogmatic about the values of the persistent parameters in the game.

5 Conclusion

The main message of this paper is that whether a misperception in the stage game is likely to survive can depend on whether agents have dogmatic or flexible views about other aspects of the stage game. In particular, we have shown that the welfare implications of an error in higher-order beliefs depend crucially on whether people know price elasticity with certainty or estimate this elasticity from past prices. Working in a canonical linear-quadratic-normal game setting, we view our paper as illustrating the practical value of the evolutionary framework in terms of guiding our thinking about the viability of biases. More broadly, our results point out that the viability of a given error must be evaluated in the context of other factors, such as whether agents engage in inference about the stage-game parameters. It may be worthwhile to investigate other factors that can enhance or hinder the viability of certain behavioral biases in future work.

Appendix

A Proof of Lemma 1

Proof. For $i \neq j$, rewrite $s_i = \left(\omega + \frac{\kappa}{\sqrt{\kappa^2+(1-\kappa)^2}} z\right) + \frac{1-\kappa}{\sqrt{\kappa^2+(1-\kappa)^2}} \eta_i$ and $s_j = \left(\omega + \frac{\kappa}{\sqrt{\kappa^2+(1-\kappa)^2}} z\right) + \frac{1-\kappa}{\sqrt{\kappa^2+(1-\kappa)^2}} \eta_j$. Note that $\omega + \frac{\kappa}{\sqrt{\kappa^2+(1-\kappa)^2}} z$ has a normal distribution with mean 0 and variance $\sigma_\omega^2 + \frac{\kappa^2}{\kappa^2+(1-\kappa)^2}\sigma_\epsilon^2$. The posterior distribution of $\left(\omega + \frac{\kappa}{\sqrt{\kappa^2+(1-\kappa)^2}} z\right)$ given s_i is therefore normal with a mean of $\frac{1/(\frac{(1-\kappa)^2}{\kappa^2+(1-\kappa)^2}\sigma_\epsilon^2)}{1/(\sigma_\omega^2+\frac{\kappa^2}{\kappa^2+(1-\kappa)^2}\sigma_\epsilon^2)+1/(\frac{(1-\kappa)^2}{\kappa^2+(1-\kappa)^2}\sigma_\epsilon^2)} s_i$ and a variance of $\frac{1}{1/(\sigma_\omega^2+\frac{\kappa^2}{\kappa^2+(1-\kappa)^2}\sigma_\epsilon^2)+1/(\frac{(1-\kappa)^2}{\kappa^2+(1-\kappa)^2}\sigma_\epsilon^2)}$.

Since η_j is mean-zero and independent of i's signal, the posterior distribution of $s_j \mid s_i$ under the correlation parameter κ is normal with a mean of

$$\frac{1/(\frac{(1-\kappa)^2}{\kappa^2+(1-\kappa)^2}\sigma_\epsilon^2)}{1/(\sigma_\omega^2+\frac{\kappa^2}{\kappa^2+(1-\kappa)^2}\sigma_\epsilon^2)+1/(\frac{(1-\kappa)^2}{\kappa^2+(1-\kappa)^2}\sigma_\epsilon^2)} s_i$$

and a variance of $\frac{1}{1/(\sigma_\omega^2+\frac{\kappa^2}{\kappa^2+(1-\kappa)^2}\sigma_\epsilon^2)+1/(\frac{(1-\kappa)^2}{\kappa^2+(1-\kappa)^2}\sigma_\epsilon^2)} + \frac{(1-\kappa)^2}{\kappa^2+(1-\kappa)^2}\sigma_\epsilon^2$. We thus define $\psi(\kappa) := \frac{1/(\frac{(1-\kappa)^2}{\kappa^2+(1-\kappa)^2}\sigma_\epsilon^2)}{1/(\sigma_\omega^2+\frac{\kappa^2}{\kappa^2+(1-\kappa)^2}\sigma_\epsilon^2)+1/(\frac{(1-\kappa)^2}{\kappa^2+(1-\kappa)^2}\sigma_\epsilon^2)}$ for $\kappa \in [0,1)$, and $\psi(1) := 1$. To see that $\psi(\kappa)$ is strictly increasing in κ, we have

$$\begin{aligned} 1/\psi(\kappa) &= 1 + \frac{\frac{(1-\kappa)^2}{\kappa^2+(1-\kappa)^2}\sigma_\epsilon^2}{\sigma_\omega^2 + \frac{\kappa^2}{\kappa^2+(1-\kappa)^2}\sigma_\epsilon^2} \\ &= 1 + \frac{(1-\kappa)^2\sigma_\epsilon^2}{(\kappa^2+(1-\kappa)^2)\sigma_\omega^2 + \kappa^2\sigma_\epsilon^2} \end{aligned}$$

and then we can verify that the second term is decreasing in κ.

As $\kappa \to 1$, the term $1/(\frac{(1-\kappa)^2}{\kappa^2+(1-\kappa)^2}\sigma_\epsilon^2)$ tends to ∞, so $\frac{1/(\frac{(1-\kappa)^2}{\kappa^2+(1-\kappa)^2}\sigma_\epsilon^2)}{1/(\sigma_\omega^2+\frac{\kappa^2}{\kappa^2+(1-\kappa)^2}\sigma_\epsilon^2)+1/(\frac{(1-\kappa)^2}{\kappa^2+(1-\kappa)^2}\sigma_\epsilon^2)}$ approaches $\frac{1/(\frac{(1-\kappa)^2}{\kappa^2+(1-\kappa)^2}\sigma_\epsilon^2)}{1/(\frac{(1-\kappa)^2}{\kappa^2+(1-\kappa)^2}\sigma_\epsilon^2)} = 1$. We also verify that $\psi(0) = \frac{1/\sigma_\epsilon^2}{(1/\sigma_\omega^2)+(1/\sigma_\epsilon^2)} > 0$.

Finally, for any $\kappa \in [0,1]$, $\frac{\kappa}{\sqrt{\kappa^2+(1-\kappa)^2}} z + \frac{1-\kappa}{\sqrt{\kappa^2+(1-\kappa)^2}} \eta_i$ has variance σ_ϵ^2 and mean 0, so $\mathbb{E}_\kappa[\omega \mid s_i] = \frac{1/\sigma_\epsilon^2}{1/\sigma_\epsilon^2+1/\sigma_\omega^2} s_i$. We then define γ as the strictly positive constant $\frac{1/\sigma_\epsilon^2}{1/\sigma_\epsilon^2+1/\sigma_\omega^2}$. □

B Proof of Lemma 2

Proof. Player i's conditional expected utility given signal s_i is

$$\begin{aligned}&\alpha_i s_i \cdot \mathbb{E}_\kappa[\omega - \frac{1}{2}r\alpha_i s_i - \frac{1}{2}r\alpha_{-i}s_{-i} + \zeta|s_i] - \frac{1}{2}(\alpha_i s_i)^2 \\ =&\alpha_i s_i \cdot (\gamma s_i - \frac{1}{2}r\alpha_i s_i - \frac{1}{2}r\psi(\kappa)s_i\alpha_{-i}) - \frac{1}{2}(\alpha_i s_i)^2 \\ =&s_i^2 \cdot (\alpha_i\gamma - \frac{1}{2}r\alpha_i^2 - \frac{1}{2}r\psi(\kappa)\alpha_i\alpha_{-i} - \frac{1}{2}\alpha_i^2).\end{aligned}$$

The term in parenthesis does not depend on s_i, and the second moment of s_i is the same for all values of κ. Therefore this expectation is $\mathbb{E}[s_i^2] \cdot \left(\alpha_i\gamma - \frac{1}{2}r\alpha_i^2 - \frac{1}{2}r\psi(\kappa)\alpha_i\alpha_{-i} - \frac{1}{2}\alpha_i^2\right)$. The expression for $\alpha_i^{BR}(\alpha_{-i};\kappa,r)$ follows from simple algebra, noting that $\mathbb{E}[s_i^2] > 0$ while the second derivative with respect to α_i for the term in the parenthesis is $-\frac{1}{2}r - \frac{1}{2} < 0$.

To see that the said linear strategy is optimal among all strategies, suppose i instead chooses any q_i after s_i. By the above arguments, the objective to maximize is

$$q_i \cdot (\gamma s_i - \frac{1}{2}rq_i - \frac{1}{2}r\psi(\kappa)s_i\alpha_{-i}) - \frac{1}{2}q_i^2.$$

This objective is a strictly concave function in q_i, as $-\frac{1}{2}r - \frac{1}{2} < 0$. The first-order condition determines the maximizer, $q_i^* = \alpha_i^{BR}(\alpha_{-i};\kappa,r) \cdot s_i$. Therefore, the linear strategy also maximizes interim expected utility after every signal s_i, so it cannot be improved upon by any other strategy. □

C Proof of Lemma 3

Proof. Conditional on the signal s_i, the distribution of market price under the model $F_{\kappa,\hat{r},\hat{\sigma}_\zeta^2}$ is normal with a mean of

$$\mathbb{E}[\omega \mid s_i] - \frac{1}{2}\hat{r}\alpha_i s_i - \frac{1}{2}\hat{r}\alpha_{-i} \cdot \mathbb{E}_\kappa[s_{-i} \mid s_i] = \gamma s_i - \frac{1}{2}\hat{r}\alpha_i s_i - \frac{1}{2}\hat{r}\alpha_{-i}\psi(\kappa)s_i,$$

while the distribution of market price under $F_{\kappa^\bullet,r^\bullet,(\sigma_\zeta^\bullet)^2}$ is normal with a mean of

$$\mathbb{E}[\omega \mid s_i] - \frac{1}{2}r^\bullet\alpha_i s_i - \frac{1}{2}r^\bullet\alpha_{-i} \cdot \mathbb{E}_{\kappa^\bullet}[s_{-i} \mid s_i] = \gamma s_i - \frac{1}{2}r^\bullet\alpha_i s_i - \frac{1}{2}r^\bullet\alpha_{-i}\psi(\kappa^\bullet)s_i.$$

Matching coefficients on s_i, we find that if $\hat{r} = r^\bullet \frac{\alpha_i+\alpha_{-i}\psi(\kappa^\bullet)}{\alpha_i+\alpha_{-i}\psi(\kappa)}$, then these means match after every s_i for any α_i, α_{-i}. On the other hand, for any other value of $\hat{r}$, these means will not match for any $s_i \neq 0$.

Conditional on the signal s_i, the variance of market price under $F_{\kappa, r^\bullet \frac{\alpha_i+\alpha_{-i}\psi(\kappa^\bullet)}{\alpha_i+\alpha_{-i}\psi(\kappa)}, \hat{\sigma}_\zeta^2}$ is

$$\mathrm{Var}_\kappa\left[\omega - \frac{1}{2} r^\bullet \frac{\alpha_i + \alpha_{-i}\psi(\kappa^\bullet)}{\alpha_i + \alpha_{-i}\psi(\kappa)} \alpha_{-i} s_{-i} \mid s_i\right] + \hat{\sigma}_\zeta^2.$$

By properties of the multivariate normal distribution, this conditional variance is constant in s_i. Let $L = \max_{\kappa \in [0,1], 0 \leq \alpha_i, \alpha_{-i} \leq \gamma} \mathrm{Var}_\kappa\left[\omega - \frac{1}{2} r^\bullet \frac{\alpha_i+\alpha_{-i}\psi(\kappa^\bullet)}{\alpha_i+\alpha_{-i}\psi(\kappa)} \alpha_{-i} s_{-i} \mid s_i\right]$. This maximum exists and is finite since the expression is a continuous function of $\kappa, \alpha_i, \alpha_{-i}$ on the compact domain $[0,1] \times [0,\gamma]^2$. The conditional variance of market price under $F_{\kappa, r^\bullet \frac{\alpha_i+\alpha_{-i}\psi(\kappa^\bullet)}{\alpha_i+\alpha_{-i}\psi(\kappa)}, \hat{\sigma}_\zeta^2}$ is bounded by $L + \hat{\sigma}_\zeta^2$ whenever $0 \leq \alpha_i, \alpha_{-i} \leq \gamma$.

On the other hand, the variance of market price under $F_{\kappa^\bullet, r^\bullet, \sigma_\zeta^\bullet}$ is at least $(\sigma_\zeta^\bullet)^2$. Thus, whenever $(\sigma_\zeta^\bullet)^2 \geq L$, there exists a unique value of $\hat{\sigma}_\zeta^2$ such that the conditional variance under $F_{\kappa, r^\bullet \frac{\alpha_i+\alpha_{-i}\psi(\kappa^\bullet)}{\alpha_i+\alpha_{-i}\psi(\kappa)}, \hat{\sigma}_\zeta^2}$ is the same as that under $F_{\kappa^\bullet, r^\bullet, (\sigma_\zeta^\bullet)^2}$ given every s_i. □

D Proof of Proposition 1

Proof. Take L as in Lemma 3. Consider a candidate linear equilibrium with strategies $0 \leq \alpha_{R\to R}, \alpha_{E\to R}, \alpha_{R\to E} \leq \gamma$, together with the self-confirming inferences given these strategies, which exist and are unique by Lemma 3. Since residents are correctly specified, it is easy to see that the only self-confirming inferences for them are $r^\bullet, (\sigma_\zeta^\bullet)^2$.

Using the equilibrium belief of the resident, we must have $\alpha_{R\to R} = \alpha_i^{BR}(\alpha_{R\to R}; \kappa^\bullet, r^\bullet)$, so $\alpha_{R\to R} = \frac{\gamma - \frac{1}{2} r^\bullet \psi(\kappa^\bullet) \alpha_{R\to R}}{1+r^\bullet}$. We find the unique solution $\alpha_{R\to R} = \frac{\gamma}{1+r^\bullet+\frac{1}{2}r^\bullet\psi(\kappa^\bullet)}$. Next, we turn to $\alpha_{R\to E}, \alpha_{E\to R}$, and r_E, the entrant's self-confirming inference. For agents in each group to best respond to each others' play and for the entrant's inferences to be self-confirming, we must have $\alpha_{R\to E} = \frac{\gamma - \frac{1}{2} r^\bullet \psi(\kappa^\bullet) \alpha_{E\to R}}{1+r^\bullet}$, $\alpha_{E\to R} = \frac{\gamma - \frac{1}{2} r_E \psi(\kappa) \alpha_{R\to E}}{1+r_E}$, and $r_E = r^\bullet \frac{\alpha_{E\to R}+\alpha_{R\to E}\psi(\kappa^\bullet)}{\alpha_{E\to R}+\alpha_{R\to E}\psi(\kappa)}$ from Lemma 3. We may rearrange the expression for $\alpha_{E\to R}$ to say $\alpha_{E\to R} = \gamma - r_E \alpha_{E\to R} - \frac{1}{2} r_E \psi(\kappa) \alpha_{R\to E}$.

Substituting the expression of r_E into this expression of $\alpha_{E\to R}$, we get

$$\begin{aligned}\alpha_{E\to R} &= \gamma - r_E \cdot (\alpha_{E\to R} + \alpha_{R\to E}\psi(\kappa) - \frac{1}{2}\alpha_{R\to E}\psi(\kappa)) \\ &= \gamma - \frac{r^\bullet \alpha_{E\to R} + r^\bullet \alpha_{R\to E}\psi(\kappa^\bullet)}{\alpha_{E\to R} + \alpha_{R\to E}\psi(\kappa)} \cdot (\alpha_{E\to R} + \alpha_{R\to E}\psi(\kappa) - \frac{1}{2}\alpha_{R\to E}\psi(\kappa)) \\ &= \gamma - r^\bullet \alpha_{E\to R} - r^\bullet \alpha_{R\to E}\psi(\kappa^\bullet) + \frac{1}{2}\psi(\kappa)\alpha_{R\to E}\frac{r^\bullet \alpha_{E\to R} + r^\bullet \alpha_{R\to E}\psi(\kappa^\bullet)}{\alpha_{E\to R} + \alpha_{R\to E}\psi(\kappa)}\end{aligned}$$

Multiply by $\alpha_{E\to R} + \alpha_{R\to E}\psi(\kappa)$ on both sides and collect terms,

$$\begin{aligned}(\alpha_{E\to R})^2 \cdot [-1 - r^\bullet] + (\alpha_{E\to R}\alpha_{R\to E}) \cdot [-\psi(\kappa) - \frac{1}{2}r^\bullet\psi(\kappa) - r^\bullet\psi(\kappa^\bullet)] \\ - (\alpha_{R\to E})^2 \cdot [\frac{1}{2}r^\bullet\psi(\kappa^\bullet)\psi(\kappa)] + \gamma[\alpha_{E\to R} + \alpha_{R\to E}\psi(\kappa)] = 0. \quad (1)\end{aligned}$$

Consider the following quadratic function in x,

$$H(x) := x^2\,[-1 - r^\bullet] + (x \cdot \ell(x)) \cdot [-\psi(\kappa) - \frac{1}{2}r^\bullet\psi(\kappa) - r^\bullet\psi(\kappa^\bullet)] - (\ell(x))^2 \cdot [\frac{1}{2}r^\bullet\psi(\kappa^\bullet)\psi(\kappa)] + \gamma\,[x + \ell(x)\psi(\kappa)] = 0, \quad (2)$$

where $\ell(x) := \frac{\gamma - \frac{1}{2}r^\bullet\psi(\kappa^\bullet)x}{1+r^\bullet}$ is a linear function in x. In a linear equilibrium, $\alpha_{E\to R}$ is a root of $H(x)$ in $[0, \frac{\gamma}{\frac{1}{2}r^\bullet\psi(\kappa^\bullet)}]$. To see why, if we were to have $\alpha_{E\to R} > \frac{\gamma}{\frac{1}{2}r^\bullet\psi(\kappa^\bullet)}$, then $\alpha_{R\to E} = 0$. In that case, $r_E = r^\bullet$ and so $\alpha_{E\to R} = \alpha_i^{BR}(0;\kappa^\bullet, r^\bullet) = \frac{\gamma}{1+r^\bullet}$. Yet $\frac{\gamma}{1+r^\bullet} < \frac{\gamma}{\frac{1}{2}r^\bullet\psi(\kappa^\bullet)}$, contradiction. Conversely, for any root x^* of $H(x)$ in $[0, \frac{\gamma}{\frac{1}{2}r^\bullet\psi(\kappa^\bullet)}]$, there is a linear equilibrium where $\alpha_{E\to R} = x^*$, $\alpha_{R\to E} = \ell(x^*) \in [0, \gamma]$, and $r_E = r^\bullet \frac{\alpha_{E\to R} + \alpha_{R\to E}\psi(\kappa^\bullet)}{\alpha_{E\to R} + \alpha_{R\to E}\psi(\kappa)}$.

Claim A.1. There exist some $\underline{\kappa}_1 < \kappa^\bullet < \bar{\kappa}_1$ so that H has a unique root in $[0, \frac{\gamma}{\frac{1}{2}r^\bullet\psi(\kappa^\bullet)}]$ for all $\kappa \in [\underline{\kappa}_1, \bar{\kappa}_1] \cap [0, 1]$.

By Claim A.1 (proved below), for $\kappa \in [\underline{\kappa}_1, \bar{\kappa}_1] \cap [0, 1]$, there is a unique linear equilibrium, where equilibrium behavior is given as a function of κ by $\alpha_{R\to R}(\kappa), \alpha_{R\to E}(\kappa)$ and $\alpha_{E\to R}(\kappa)$.

Recall from Lemma 2 that the objective expected utility from playing α_i against an opponent who plays α_{-i} is $U^\bullet(\alpha_i, \alpha_{-i}) = \mathbb{E}[s_i^2] \cdot (\alpha_i\gamma - \frac{1}{2}r^\bullet\alpha_i^2 - \frac{1}{2}r^\bullet\psi(\kappa^\bullet)\alpha_i\alpha_{-i} - \frac{1}{2}\alpha_i^2)$. If $-i$ plays the rational best response, then the objective expected utility of choosing α_i is $\bar{U}_i(\alpha_i) := \mathbb{E}[s_i^2] \cdot (\alpha_i\gamma - \frac{1}{2}r^\bullet\alpha_i^2 - \frac{1}{2}r^\bullet\psi(\kappa^\bullet)\alpha_i\frac{\gamma - \frac{1}{2}r^\bullet\psi(\kappa^\bullet)\alpha_i}{1+r^\bullet} - \frac{1}{2}\alpha_i^2)$. The derivative in α_i is $\bar{U}_i'(\alpha_i) = \gamma - r^\bullet\alpha_i - \frac{1}{2}\frac{r^\bullet}{1+r^\bullet}\gamma\psi(\kappa^\bullet) + \frac{1}{2}\frac{(r^\bullet)^2\psi(\kappa^\bullet)^2}{1+r^\bullet}\alpha_i - \alpha_i$. We also know that $\alpha_{R\to R} = \frac{\gamma}{1+r^\bullet+\frac{1}{2}r^\bullet\psi(\kappa^\bullet)}$

satisfies the first-order condition that $\gamma - r^\bullet \alpha_{R\to R} - \frac{1}{2} r^\bullet \psi(\kappa^\bullet)\alpha_{R\to R} - \alpha_{R\to R} = 0$, therefore

$$\begin{aligned}\bar{U}_i'(\alpha_{R\to R}) &= -\frac{1}{2}\frac{r^\bullet}{1+r^\bullet}\gamma\psi(\kappa^\bullet) + \frac{1}{2}\frac{(r^\bullet)^2\psi(\kappa^\bullet)^2}{1+r^\bullet}\alpha_{R\to R} + \frac{1}{2}r^\bullet\psi(\kappa^\bullet)\alpha_{R\to R} \\ &= \left[\frac{r^\bullet\psi(\kappa^\bullet)}{2}\right]\left(\frac{-\gamma}{1+r^\bullet} + \frac{\alpha_{R\to R}\psi(\kappa^\bullet)r^\bullet}{1+r^\bullet} + \alpha_{R\to R}\right).\end{aligned}$$

Making the substitution $\alpha_{R\to R} = \frac{\gamma}{1+r^\bullet+\frac{1}{2}r^\bullet\psi(\kappa^\bullet)}$,

$$\begin{aligned}\frac{-\gamma}{1+r^\bullet} + \frac{\alpha_{R\to R}\psi(\kappa^\bullet)r^\bullet}{1+r^\bullet} + \alpha_{R\to R} &= \frac{-\gamma(1+r^\bullet+\frac{1}{2}\psi(\kappa^\bullet)r^\bullet) + \gamma\psi(\kappa^\bullet)r^\bullet + \gamma(1+r^\bullet)}{(1+r^\bullet)(1+r^\bullet+\frac{1}{2}\psi(\kappa^\bullet)r^\bullet)} \\ &= \frac{\frac{1}{2}\gamma\psi(\kappa^\bullet)r^\bullet}{(1+r^\bullet)(1+r^\bullet+\frac{1}{2}\psi(\kappa^\bullet)r^\bullet)} > 0.\end{aligned}$$

Therefore, if we can show that $\alpha'_{E\to R}(\kappa^\bullet) > 0$, then there exists some $\underline{\kappa}_1 \leq \underline{\kappa} < \kappa^\bullet < \bar{\kappa} \leq \bar{\kappa}_1$ so that for every $\kappa \in [\underline{\kappa}, \bar{\kappa}] \cap [0,1]$, $\kappa \neq \kappa^\bullet$ entrants have strictly higher or strictly lower equilibrium fitness in the unique linear equilibrium than residents, depending on the sign of $\kappa - \kappa^\bullet$. Consider again the quadratic function $H(x)$ in Equation (2) and implicitly characterize the unique root x in $[0, \frac{\gamma}{\frac{1}{2}r^\bullet\psi(\kappa^\bullet)}]$ as a function of κ in a neighborhood around $\kappa^\bullet$. Denote this root by α^M, let $D := \frac{d\alpha^M}{d\psi(\kappa)}$ and also note $\frac{d\ell(\alpha^M)}{d\psi(\kappa)} = \frac{-r^\bullet}{2(1+r^\bullet)}\psi(k^\bullet)\cdot D$. We have

$$\begin{aligned}&(-1-r^\bullet)\cdot(2\alpha^M)\cdot D + (\alpha^M\ell(\alpha^M))(-1-\frac{1}{2}r^\bullet) \\ &+ (\ell(\alpha^M)D + \alpha^M\frac{-r^\bullet}{2(1+r^\bullet)}\psi(\kappa^\bullet)D)\cdot(-\psi(\kappa) - \frac{1}{2}r^\bullet\psi(\kappa) - r^\bullet\psi(\kappa^\bullet)) + (\ell(\alpha^M))^2\cdot(-\frac{1}{2}r^\bullet\psi(\kappa^\bullet)) \\ &+ (2\ell(\alpha^M)\frac{-r^\bullet}{2(1+r^\bullet)}\psi(\kappa^\bullet)D)\cdot(-\frac{1}{2}r^\bullet\psi(\kappa^\bullet)\psi(\kappa)) + \gamma(D + \ell(\alpha^M) + \psi(\kappa)\frac{-r^\bullet}{2(1+r^\bullet)}\psi(\kappa^\bullet)D) = 0\end{aligned}$$

Evaluate at $\kappa = \kappa^\bullet$, noting that $\alpha^M(\kappa^\bullet) = \ell(\alpha^M(\kappa^\bullet)) = x^* := \frac{\gamma}{1+r^\bullet+\frac{1}{2}\psi(\kappa^\bullet)r^\bullet}$. The terms without D are:

$$\begin{aligned}(x^*)^2(-1-\frac{1}{2}r^\bullet) + (x^*)^2(\frac{1}{2}r^\bullet\psi(\kappa^\bullet)) + \gamma x^* &= x^*\cdot\left[-x^*\cdot\left(1+r^\bullet+\frac{1}{2}\psi(\kappa^\bullet)r^\bullet - \frac{1}{2}r^\bullet\right) + \gamma\right] \\ &= x^*\cdot\left[-\gamma + \frac{1}{2}x^*r^\bullet + \gamma\right] = \frac{1}{2}r^\bullet(x^*)^2 > 0.\end{aligned}$$

The coefficient in front of D is:

$$(-1-r^\bullet)(2x^*) + (x^* + x^*\frac{-r^\bullet}{2(1+r^\bullet)}\psi(\kappa^\bullet))\cdot(-\psi(\kappa^\bullet) - \frac{3}{2}r^\bullet\psi(\kappa^\bullet)) + \frac{1}{2}x^*\frac{(r^\bullet)^2}{(1+r^\bullet)}\psi(\kappa^\bullet)^3 + \gamma + \gamma\psi(\kappa^\bullet)^2\cdot\frac{-r^\bullet}{2(1+r^\bullet)}.$$

Make the substitution $\gamma = x^* \cdot \Big(1 + r^\bullet + \frac{1}{2}\psi(\kappa^\bullet)r^\bullet\Big)$,

$$x^* \cdot \left\{-2 - 2r^\bullet + \left(1 - \frac{r^\bullet}{2(1+r^\bullet)}\psi(\kappa^\bullet)\right) \cdot \psi(\kappa^\bullet)(-\frac{3}{2}r^\bullet - 1) + \frac{(r^\bullet)^2}{2(1+r^\bullet)}\psi(\kappa^\bullet)^3\right\}$$
$$+x^* \cdot \left\{\left(1 + r^\bullet + \frac{1}{2}\psi(\kappa^\bullet)r^\bullet\right) \cdot (1 - \psi(\kappa^\bullet)^2\frac{r^\bullet}{2(1+r^\bullet)})\right\}.$$

Collect terms inside the parenthesis based on powers of $\psi(\kappa^\bullet)$, we get

$$x^* \cdot \left\{\psi(\kappa^\bullet)^3\frac{(r^\bullet)^2}{2(1+r^\bullet)} - \frac{\psi(\kappa^\bullet)^2 r^\bullet}{2(1+r^\bullet)}(-\frac{3}{2}r^\bullet - 1) + \psi(\kappa^\bullet)(-\frac{3}{2}r^\bullet - 1) - 2r^\bullet - 2\right\}$$
$$+x^* \cdot \left\{-\psi(\kappa^\bullet)^3\frac{(r^\bullet)^2}{4(1+r^\bullet)} - \frac{\psi(\kappa^\bullet)^2 r^\bullet}{2(1+r^\bullet)} \cdot (1+r^\bullet) + 1 + r^\bullet + \frac{1}{2}\psi(\kappa^\bullet)r^\bullet\right\}.$$

Combine to get: $x^* \cdot \Big[\psi(\kappa^\bullet)^3\frac{(r^\bullet)^2}{4(1+r^\bullet)} + \frac{\psi(\kappa^\bullet)^2(r^\bullet)^2}{4(1+r^\bullet)} - \psi(\kappa^\bullet)r^\bullet - \psi(\kappa^\bullet) - r^\bullet - 1\Big]$. Here $\psi(\kappa^\bullet)^3\frac{(r^\bullet)^2}{4(1+r^\bullet)}$ and $\frac{\psi(\kappa^\bullet)^2(r^\bullet)^2}{4(1+r^\bullet)}$ are positive terms with $\psi(\kappa^\bullet)^3\frac{(r^\bullet)^2}{4(1+r^\bullet)} + \frac{\psi(\kappa^\bullet)^2(r^\bullet)^2}{4(1+r^\bullet)} \le \frac{(r^\bullet)^2}{4(1+r^\bullet)} + \frac{(r^\bullet)^2}{4(1+r^\bullet)} \le \frac{1}{2} \cdot r^\bullet \cdot \frac{r^\bullet}{1+r^\bullet} \le \frac{1}{2}r^\bullet$. Now $-r^\bullet + \frac{1}{2} \cdot r^\bullet < 0$, and also $-\psi(\kappa^\bullet)r^\bullet - \psi(\kappa^\bullet) - 1 < 0$. Thus, the coefficient in front of D is strictly negative. This shows $D(\kappa^\bullet) > 0$. Finally, $\frac{d\alpha^M}{d\psi(\kappa)}$ has the same sign as $\frac{d\alpha^M}{d\kappa}$ since $\psi(\kappa)$ is strictly increasing in κ. □

D.1 Proof of Claim A.1

Proof. We show that $H(x)$ (i) has a unique root in $[0, \frac{\gamma}{\frac{1}{2}r^\bullet\psi(\kappa^\bullet)}]$ when $\kappa = \kappa^\bullet$; (ii) does not have a root at $x = 0$ or $x = \frac{\gamma}{\frac{1}{2}r^\bullet\psi(\kappa^\bullet)}$, and (iii) the root in the interval is not a double root. By these three statements, since $H(x)$ is a continuous function of κ, there must exist some $\underline{\kappa}_1 < \kappa^\bullet < \bar{\kappa}_1$ so that it continues to have a unique root in $[0, \frac{\gamma}{\frac{1}{2}r^\bullet\psi(\kappa^\bullet)}]$ for all $\kappa \in [\underline{\kappa}_1, \bar{\kappa}_1] \cap [0, 1]$.

Statement (i) has to do with the fact that if $\kappa = \kappa^\bullet$, then we need $\alpha_{R\to E} = \frac{\gamma - \frac{1}{2}r^\bullet\psi(\kappa^\bullet)\alpha_{E\to R}}{1+r^\bullet}$ and $\alpha_{E\to R} = \frac{\gamma - \frac{1}{2}r^\bullet\psi(\kappa^\bullet)\alpha_{R\to E}}{1+r^\bullet}$. These are linear best response functions with a slope of $-\frac{1}{2}\frac{r^\bullet}{1+r^\bullet}\psi(\kappa^\bullet)$, which falls in $(-\frac{1}{2}, 0)$. So there can only be one solution to H in that region (even when we allow $\alpha_{R\to E} \neq \alpha_{E\to R}$), which is the symmetric equilibrium found before $\alpha_{R\to E} = \alpha_{E\to R} = \frac{\gamma}{1+r^\bullet+\frac{1}{2}r^\bullet\psi(\kappa^\bullet)}$.

For Statement (ii), we evaluate $H(0) = -(\frac{\gamma}{1+r^\bullet})^2\frac{1}{2}r^\bullet\psi(\kappa^\bullet)^2 + \frac{\gamma^2\psi(\kappa^\bullet)}{1+r^\bullet} = \frac{\psi(\kappa^\bullet)\gamma^2}{1+r^\bullet}(1 - \frac{(1/2)r^\bullet\psi(\kappa^\bullet)}{1+r^\bullet}) \neq 0$ because $1+r^\bullet > (1/2)r^\bullet\psi(\kappa^\bullet)$. Finally, we evaluate $H(\frac{\gamma}{\frac{1}{2}r^\bullet\psi(\kappa^\bullet)}) = (\frac{\gamma}{\frac{1}{2}r^\bullet\psi(\kappa^\bullet)})^2(-1-r^\bullet) + \gamma\frac{\gamma}{\frac{1}{2}r^\bullet\psi(\kappa^\bullet)} = \frac{\gamma^2}{\frac{1}{2}r^\bullet\psi(\kappa^\bullet)}(1 - \frac{1+r^\bullet}{\frac{1}{2}r^\bullet\psi(\kappa^\bullet)})$. This expression is once again not 0 because $1 + r^\bullet > (1/2)r^\bullet\psi(\kappa^\bullet)$.

For Statement (iii), we show that $H^{'}(x^*) < 0$ where $x^* = \frac{\gamma}{1+r^\bullet+\frac{1}{2}r^\bullet\psi(\kappa^\bullet)}$. We find that

$$\begin{aligned}H^{'}(x) =&2x(-1-r^\bullet) + \left(\frac{\gamma - r^\bullet\psi(\kappa^\bullet)x}{1+r^\bullet}\right)(-\psi(\kappa^\bullet) - \frac{1}{2}r^\bullet\psi(\kappa^\bullet) - r^\bullet\psi(\kappa^\bullet))\\ &-2\left(\frac{\gamma - \frac{1}{2}r^\bullet\psi(\kappa^\bullet)x}{1+r^\bullet}\right)\left(\frac{-\frac{1}{2}r^\bullet\psi(\kappa^\bullet)}{1+r^\bullet}\right)\left(\frac{1}{2}r^\bullet\psi(\kappa^\bullet)^2\right) + \gamma - \frac{\frac{1}{2}r^\bullet\psi(\kappa^\bullet)}{1+r^\bullet}\gamma\psi(\kappa^\bullet).\end{aligned}$$

Collecting terms, the coefficient on x is

$$-2-2r^\bullet + \frac{\psi(\kappa^\bullet)^2 r^\bullet}{1+r^\bullet}\left(\frac{3}{2}r^\bullet + 1 - \frac{1}{4}(\frac{(r^\bullet)^2\psi(\kappa^\bullet)^2}{1+r^\bullet})\right),$$

while the coefficient on the constant is

$$\frac{\gamma\psi(\kappa^\bullet)}{1+r^\bullet}\left(-\frac{3}{2}r^\bullet - 1 + \frac{1}{2}\frac{(r^\bullet)^2\psi(\kappa^\bullet)^2}{1+r^\bullet} - \frac{1}{2}r^\bullet\psi(\kappa^\bullet)\right) + \gamma.$$

Therefore, we may calculate $H^{'}(x^*) \cdot \frac{1}{x^*}(1+r^\bullet)^2$, which has the same sign as $H^{'}(x^*)$, to be:

$$\begin{aligned}&-(1+r^\bullet)^2(2+2r^\bullet) + \psi(\kappa^\bullet)^2 r^\bullet((1+r^\bullet)(\frac{3}{2}r^\bullet+1) - \frac{1}{4}(r^\bullet)^2\psi(\kappa^\bullet)^2)\\ &+(1+r^\bullet+\frac{1}{2}r^\bullet\psi(\kappa^\bullet))\left[\psi(\kappa^\bullet)((1+r^\bullet)[-\frac{3}{2}r^\bullet - 1 - \frac{1}{2}r^\bullet\psi(\kappa^\bullet)] + \frac{1}{2}(r^\bullet)^2\psi(\kappa^\bullet)^2) + (1+r^\bullet)^2\right].\end{aligned}$$

We have

$$-(1+r^\bullet)^2(2+2r^\bullet) + (1+r^\bullet+\frac{1}{2}r^\bullet\psi(\kappa^\bullet))(1+r^\bullet)^2 \leq (1+r^\bullet)^2(-1-\frac{1}{2}r^\bullet) < 0,$$

since $0 \leq \psi(\kappa^\bullet) \leq 1$. Also, for the same reason,

$$(1+r^\bullet)[-\frac{1}{2}r^\bullet\psi(\kappa^\bullet)] + \frac{1}{2}(r^\bullet)^2\psi(\kappa^\bullet)^2 \leq -\frac{1}{2}(r^\bullet)^2\psi(\kappa^\bullet) + \frac{1}{2}(r^\bullet)^2\psi(\kappa^\bullet)^2 \leq 0.$$

Finally, $\psi(\kappa^\bullet)^2 r^\bullet(1+r^\bullet)(\frac{3}{2}r^\bullet+1) + (1+r^\bullet+\frac{1}{2}r^\bullet\psi(\kappa^\bullet))\psi(\kappa^\bullet)(1+r^\bullet)(-\frac{3}{2}r^\bullet - 1)$ is no larger than

$$\begin{aligned}&\psi(\kappa^\bullet)^2 r^\bullet(\frac{3}{2}(r^\bullet)^2 + \frac{5}{2}r^\bullet + 1) + [r^\bullet\psi(\kappa^\bullet)r^\bullet(-(3/2)r^\bullet)]\\ &+[r^\bullet\psi(\kappa^\bullet)r^\bullet(-1) + 1\cdot\psi(\kappa^\bullet)r^\bullet(-(3/2)r^\bullet)] + [r^\bullet\psi(\kappa^\bullet)\cdot 1\cdot(-1)]\end{aligned}$$

where the negative terms in the first, second, and third square brackets are respectively larger in absolute value than the first, second and third parts in the expansion of the first summand. Therefore, we conclude $H^{'}(x^*) < 0$. □

E Proof of Proposition 2

Proof. We will show that in every linear equilibrium: (i) for each $g \in \{R, E\}$, the inferred elasticity under κ_g is $\frac{1+\psi(\kappa^\bullet)}{1+\psi(\kappa_g)} r^\bullet$; (ii) for each $g \in \{R, E\}$, $\alpha_{g\to g} = \frac{\gamma}{1+\frac{r^\bullet}{2}(1+\psi(\kappa^\bullet))+\frac{r^\bullet}{2}(\frac{1+\psi(\kappa^\bullet)}{1+\psi(\kappa_g)})}$; (iii) the equilibrium fitness of group g is weakly higher than that of group g' if and only if $\kappa_g \le \kappa_{g'}$.

Take L as in Lemma 3. In any linear equilibrium, by Lemma 3, group g agents infer elasticity $r_i^{INF}(\alpha_{g\to g}, \alpha_{g\to g}; \kappa^\bullet, \kappa_g, r^\bullet) = \frac{\alpha_{g\to g}+\alpha_{g\to g}\psi(\kappa^\bullet)}{\alpha_{g\to g}+\alpha_{g\to g}\psi(\kappa_g)} r^\bullet = \frac{1+\psi(\kappa^\bullet)}{1+\psi(\kappa_g)} r^\bullet$, proving (i).

Given this belief, we must have $\alpha_{g\to g} = \frac{\gamma - \frac{1}{2}\frac{1+\psi(\kappa^\bullet)}{1+\psi(\kappa_g)} r^\bullet \psi(\kappa_g)\alpha_{g\to g}}{1+\frac{1+\psi(\kappa^\bullet)}{1+\psi(\kappa_g)} r^\bullet}$ by Lemma 2. Rearranging yields $\alpha_{g\to g} = \frac{\gamma}{1+\frac{r^\bullet}{2}(1+\psi(\kappa^\bullet))+\frac{r^\bullet}{2}(\frac{1+\psi(\kappa^\bullet)}{1+\psi(\kappa)})}$, proving (ii).

From Lemma 2, the objective expected utility of each player when both play the strategy profile α_{symm} is $\mathbb{E}[s_i^2] \cdot \left(\alpha_{symm}\gamma - \frac{1}{2} r^\bullet \alpha_{symm}^2 - \frac{1}{2} r^\bullet \psi(\kappa^\bullet)\alpha_{symm}^2 - \frac{1}{2}\alpha_{symm}^2\right)$. This function is strictly concave and quadratic in α_{symm} that is 0 at $\alpha_{symm} = 0$. Therefore, it is strictly decreasing in α_{symm} for α_{symm} larger than the team solution α_{TEAM} that maximizes this expression, given by the first-order condition

$$\gamma - r^\bullet \alpha_{TEAM} - r^\bullet \psi(\kappa^\bullet)\alpha_{TEAM} - \alpha_{TEAM} = 0 \Rightarrow \alpha_{TEAM} = \frac{\gamma}{1 + r^\bullet + r^\bullet \psi(\kappa^\bullet)}.$$

For any value of $\kappa \in [0, 1]$, using the fact that $\psi(0) > 0$ and ψ is strictly increasing,

$$\frac{\gamma}{1 + \frac{r^\bullet}{2}(1 + \psi(\kappa^\bullet)) + \frac{r^\bullet}{2}(\frac{1+\psi(\kappa^\bullet)}{1+\psi(\kappa)})} > \frac{\gamma}{1 + \frac{r^\bullet}{2}(1 + \psi(\kappa^\bullet)) + \frac{r^\bullet}{2}(1 + \psi(\kappa^\bullet))} = \alpha_{TEAM}.$$

Also, $\frac{\gamma}{1+\frac{r^\bullet}{2}(1+\psi(\kappa^\bullet))+\frac{r^\bullet}{2}(\frac{1+\psi(\kappa^\bullet)}{1+\psi(\kappa)})}$ is a strictly increasing function in κ, since ψ is strictly increasing. We therefore conclude that each player's utility when they play $\frac{\gamma}{1+\frac{r^\bullet}{2}(1+\psi(\kappa^\bullet))+\frac{r^\bullet}{2}(\frac{1+\psi(\kappa^\bullet)}{1+\psi(\kappa)})}$ against each other is strictly decreasing in κ, proving (iii). □

E.1 Proof of Proposition 3

Proof. Take L as in Lemma 3. Suppose the entrant has the dogmatic model with fixed parameters $\kappa, r^\bullet, (\sigma_\zeta^\bullet)^2$ for some $\kappa \in [0,1]$. Following the same steps of the proof of Proposition 1, there exists exactly one linear equilibrium, and it involves residents playing $\frac{\gamma}{1+r^\bullet+\frac{1}{2}r^\bullet\psi(\kappa^\bullet)}$ against each other.

We now analyze $\alpha_{E\to R}(\kappa)$ in such linear equilibrium with uniform matching. In the proof of Proposition 1, we defined $\bar{U}_i(\alpha_i)$ as i's objective expected utility of choosing α_i when $-i$ plays the rational best response. We showed that $\bar{U}_i'(\frac{\gamma}{1+r^\bullet+\frac{1}{2}r^\bullet\psi(\kappa^\bullet)}) > 0$. In a linear equilibrium where i believes in parameter $(\kappa, r^\bullet, (\sigma_\zeta^\bullet)^2)$ and $-i$ believes in parameters $(\kappa^\bullet, r^\bullet, (\sigma_\zeta^\bullet)^2)$, using the expression for α_i^{BR} from Lemma 2, the play of i solves $x = \frac{\gamma-\frac{1}{2}r^\bullet\psi(\kappa)\left(\frac{\gamma-\frac{1}{2}r^\bullet\psi(\kappa^\bullet)x}{1+r^\bullet}\right)}{1+r^\bullet}$, which implies $\alpha_{E\to R}(\kappa) = \frac{\gamma(1+r^\bullet-\frac{1}{2}\psi(\kappa)r^\bullet)}{1+2r^\bullet+(r^\bullet)^2-\frac{1}{4}\psi(\kappa)\psi(\kappa^\bullet)(r^\bullet)^2}$. Taking the derivative and evaluating at $\kappa = \kappa^\bullet$, we find an expression with the same sign as $\frac{1}{4}\psi'(\kappa^\bullet)r^\bullet(1+r^\bullet)\gamma(-2(1+r^\bullet)+\psi(\kappa^\bullet)r^\bullet)$, which is strictly negative because $\psi'(\kappa^\bullet) > 0$, $r^\bullet > 0$, $\gamma > 0$, and $\psi(\kappa^\bullet) \le 1$. This shows there exists $\epsilon > 0$ so that for every $\kappa_h \in (\kappa^\bullet, \kappa^\bullet+\epsilon]$, we have $\bar{U}_i(\alpha_{E\to R}(\kappa_h)) < \bar{U}_i(\frac{\gamma}{1+r^\bullet+\frac{1}{2}r^\bullet\psi(\kappa^\bullet)})$, that is entrants with $\kappa_E = \kappa_h$ have strictly lower fitness than residents with $\kappa_R = \kappa^\bullet$ in the unique linear equilibrium for uniform matching. This argument establishes the first claim.

Next, we turn to $\alpha_{E\to E}(\kappa)$ with assortative matching. Using the expression for α_i^{BR} in Lemma 2, we find that $\alpha_{E\to E}(\kappa) = \frac{\gamma}{1+r^\bullet+\frac{1}{2}r^\bullet\psi(\kappa)}$. Since $\psi' > 0$, we have $\alpha_{E\to E}(\kappa)$ is strictly larger than $\alpha_{R\to R} = \frac{\gamma}{1+r^\bullet+\frac{1}{2}r^\bullet\psi(\kappa^\bullet)}$ when $\kappa < \kappa^\bullet$. From the proof of Proposition 2, we know that objective payoffs in the stage game are strictly decreasing in linear strategies larger than the team solution $\alpha_{TEAM} = \frac{\gamma}{1+r^\bullet+r^\bullet\psi(\kappa^\bullet)}$. Since $\alpha_{E\to E}(\kappa) > \alpha_{R\to R} > \alpha_{TEAM}$, we conclude the entrants with $\kappa_E = \kappa_l$ have strictly lower fitness than residents with $\kappa_R = \kappa^\bullet$ in the unique linear equilibrium with assortative matching for any $\kappa_l < \kappa^\bullet$. This argument establishes the second claim. □

Supplemental Appendix for "Higher-Order Beliefs and (Mis)learning from Prices"

Kevin He and Jonathan Libgober

SA 1 More General LQN Games

We turn to general incomplete-information games and provide a condition for a model to be susceptible to invasion from a "nearby" misspecified model. This condition shows how assortativity and the learning channel shape the evolutionary selection of models for a broader class of stage games and biases. We also relate the condition to the results studied in the main text.

Consider a stage game where a state of the world ω is drawn each time the game is played. Players 1 and 2 observe private signals $s_1, s_2 \in S \subseteq \mathbb{R}$, possibly correlated given ω. The objective distribution of (ω, s_1, s_2) is $\mathbb{P}^\bullet$. Based on their signals, players choose actions $q_1, q_2 \in \mathbb{R}$ and receive random consequences $y_1, y_2 \in \mathbb{Y}$. The distribution over consequences as a function of $(\omega, s_1, s_2, q_1, q_2)$ and the utility over consequences $\pi : \mathbb{Y} \to \mathbb{R}$ are such that each player i's objective expected utility from taking action q_i against opponent action q_{-i} in state ω is given by $u_i^\bullet(q_i, q_{-i}; \omega)$, differentiable in its first two arguments.

For an interval of real numbers $[\underline{\kappa}, \bar{\kappa}]$ with $\underline{\kappa} < \bar{\kappa}$ and $\kappa^\bullet \in (\underline{\kappa}, \bar{\kappa})$, suppose there is a family of models $(\Theta(\kappa))_{\kappa \in [\underline{\kappa}, \bar{\kappa}]}$. Let $\lambda = 0$ denote the case where matching is uniform and $\lambda = 1$ denote the case where it is assortative. We fix a strategy space $\mathbb{A} \subseteq \mathbb{R}^S$, representing the feasible signal-contingent strategies. Suppose the resident model in the society is $\Theta_R = \Theta(\kappa^\bullet)$, and the entrant model is $\Theta_E = \Theta(\kappa)$ for some $\kappa \in [\underline{\kappa}, \bar{\kappa}]$. Each model Θ_g may be flexible, in which case it contains a set of parameters whose values must be inferred to match the observable distribution over consequences given the equilibrium strategies. Alternatively, Θ_g may be dogmatic, in which case it fixes the agent's belief about the values of all persistent parameters in the game.

As in Definition 4, a *linear equilibrium* for a given interaction structure $\lambda \in \{0, 1\}$ consists of each group's beliefs about the persistent game parameters and each group's strategy when playing each type of opponent, such that:

- Every agent maximizes their subjective expected utility in every match (given opponent's strategy and their belief about the persistent game parameters),

- Beliefs are either the fixed parameters in a group's dogmatic model or self-confirming inferences of the free parameters in a group's flexible model — that is, parameters values that generated the observed distribution of consequences in the matches that the group faces with probability 1, and

- All strategies are linear functions of private signal realizations – that is, for every group g and every opponent group g', the strategy $\sigma_{g\to g'}(s_i) = \alpha_{g\to g'}(s_i) \cdot s_i$ for some real number $\alpha_{g\to g'}(s_i)$.

The next assumption requires there to be a unique linear equilibrium for either interaction structure and for any $\kappa \in [\underline{\kappa}, \bar{\kappa}]$. A significant portion of our analysis for our duopoly model aimed to demonstrate that these existence and uniqueness properties held. On the other hand, linear equilibria exist and are unique in a large class of games outside of the duopoly framework, and in particular in LQN games under some conditions on the payoff functions (see, e.g., Angeletos and Pavan (2007)), and hence we do not expect them to be particularly sensitive to specific details of this environment.

Assumption SA1. *Suppose there is a unique linear equilibrium for either* $\lambda \in \{0, 1\}$*, with* $\Theta_R = \Theta(\kappa^\bullet)$*,* $\Theta_E = \Theta(\kappa)$ *for every* $\kappa \in [\underline{\kappa}, \bar{\kappa}]$*. Suppose the* κ*-indexed linear equilibrium strategies coefficients* $\alpha_{R\to R}(\kappa), \alpha_{R\to E}(\kappa), \alpha_{E\to R}(\kappa), \alpha_{R\to R}(\kappa)$ *are differentiable in* κ*. Finally, suppose that in the linear equilibrium with* $\kappa = \kappa^\bullet$*,* $\alpha_{R\to R}(\kappa^\bullet)$ *is objectively interim-optimal against itself.*[5]

Proposition SA1. *Fix* $\lambda \in \{0, 1\}$ *and let* $\alpha^\bullet := \alpha_{R\to R}(\kappa^\bullet)$*. Then, under Assumption SA1, if*

$$\mathbb{E}^\bullet\left[\mathbb{E}^\bullet\left[\frac{\partial u_1^\bullet}{\partial q_2}(\alpha^\bullet s_1, \alpha^\bullet s_2, \omega) \cdot [(1-\lambda)\alpha'_{R\to E}(\kappa^\bullet) + \lambda\alpha'_{E\to E}(\kappa^\bullet)] \cdot s_2 \mid s_1\right]\right] > 0,$$

then there exists some $\epsilon > 0$ *so that* $\Theta(\kappa^\bullet)$ *is susceptible to invasion from models* $\Theta(\kappa)$ *with* $\kappa \in (\kappa^\bullet, \kappa^\bullet + \epsilon] \cap [\underline{\kappa}, \bar{\kappa}]$*, with* λ *interaction structure. Also, if*

$$\mathbb{E}^\bullet\left[\mathbb{E}^\bullet\left[\frac{\partial u_1^\bullet}{\partial q_2}(\alpha^\bullet s_1, \alpha^\bullet s_2, \omega) \cdot [(1-\lambda)\alpha'_{R\to E}(\kappa^\bullet) + \lambda\alpha'_{E\to E}(\kappa^\bullet)] \cdot s_2 \mid s_1\right]\right] < 0,$$

[5] We say $\alpha_{R\to R}$ is objectively interim-optimal against itself if, for every $s_i \in S$, $\alpha_{R\to R}(\kappa^\bullet) \cdot s_i$ maximizes the agent's objective expected utility across all of $\mathbb{R}$ when $-i$ uses the same linear strategy $\alpha_{R\to R}(\kappa^\bullet)$.

then there exists some $\epsilon > 0$ so that $\Theta(\kappa^\bullet)$ is susceptible to invasion from models $\Theta(\kappa)$ with $\kappa \in [\kappa^\bullet - \epsilon, \kappa^\bullet) \cap [\underline{\kappa}, \bar{\kappa}]$, with λ interaction structure. Here $\mathbb{E}^\bullet$ is the expectation with respect to the objective distribution of (ω, s_1, s_2) under $\mathbb{P}^\bullet$.

Proposition SA1 describes a general condition to determine whether a correctly specified model is evolutionarily fragile against a nearby misspecified entrant model. The condition asks if a slight change in the entrant model's κ leads entrants' opponents to change their equilibrium actions such that the entrants become better off on average. These opponents are the residents under uniform matching $\lambda = 0$, so $\alpha'_{R\to E}(\kappa^\bullet)$ is relevant. These opponents are other entrants under assortative matching $\lambda = 1$, so $\alpha'_{E\to E}(\kappa^\bullet)$ is relevant.

Proposition SA1 implies that one should only expect the correctly specified model to be resistant to invasion from all nearby models in "special" cases — that is, when the expectation in the statement of Proposition SA1 is exactly equal to 0. One such special case is when the agents face a decision problem where 2's action does not affect 1's payoffs, that is $\frac{\partial u_1^\bullet}{\partial q_2} = 0$. This condition sets the expectation to zero, so the result never implies that the correctly specified model is susceptible to invasion from a misspecified model in such decision problems.

In the duopoly game analyzed previously, we have $\frac{\partial u_1^\bullet}{\partial q_2}(q_1, q_2, \omega) = -\frac{1}{2}r^\bullet q_1$. Player 1 is harmed by player 2 producing more if $q_1 > 0$ and helped if $q_1 < 0$. From straightforward algebra, the expectation in Proposition SA1 simplifies to

$$\mathbb{E}^\bullet[s_1^2] \cdot (-\frac{1}{2}\psi(\kappa^\bullet) r^\bullet \alpha^\bullet) \cdot [(1-\lambda)\alpha'_{R\to E}(\kappa^\bullet) + \lambda \alpha'_{E\to E}(\kappa^\bullet)].$$

The structure of the LQN game is also used in signing the derivatives of $\alpha_{g\to g'}(\kappa)$ (although doing so may still be tractable in other environments with other kinds of structure). Focusing on the case where all agents have flexible models, the proof of Proposition 1 shows that when $\lambda = 0$, $\alpha'_{R\to E}(\kappa^\bullet) < 0$. The proof of Proposition 2 shows that when $\lambda = 1$, $\alpha'_{E\to E}(\kappa^\bullet) > 0$. The uniqueness of linear equilibrium also follows from these results, for an open interval of κ containing $\kappa^\bullet$. Lemma 2 implies linear strategies played by two correctly specified players against each other are objectively interim-optimal. So, the conditions of Proposition SA1 hold for $\lambda \in \{0, 1\}$, and we deduce that with flexible models, the correctly specified model is susceptible to invasion from slightly higher κ (for $\lambda = 0$) and slightly lower κ (for $\lambda = 1$).

SA 1.1 Proof of Proposition SA1

Proof. Consider the society where $\Theta_R = \Theta_E = \Theta(\kappa^\bullet)$, where Θ_R is the resident model and Θ_E is the entrant model. For any linear equilibrium with behavior $(\sigma_{R\to R}, \sigma_{R\to E}, \sigma_{E\to R}, \sigma_{E\to E})$ and beliefs $\theta_R \in \Theta_R$ and $\theta_E \in \Theta_E$, there exists another linear equilibrium $(\sigma'_{R\to R}, \sigma'_{R\to E}, \sigma'_{E\to R}, \sigma'_{E\to E})$ where $\sigma'_{g,g'} = \sigma_{R\to R}$ for all $g, g' \in \{A, B\}$ and all agents infer parameters θ_R. The uniqueness of linear equilibrium from Assumption SA1 implies $\alpha_{R\to E}(\kappa^\bullet) = \alpha_{E\to R}(\kappa^\bullet) = \alpha_{E\to E}(\kappa^\bullet) = \alpha_{R\to R}(\kappa^\bullet) = \alpha^\bullet$.

Now consider the society where $\Theta_E = \Theta(\kappa)$. By assumption, there exists a linear equilibrium where $\alpha_{R\to R}(\kappa) = \alpha_{R\to R}(\kappa^\bullet)$. Since we also take it to be unique, we must in fact have $\alpha_{R\to R}(\kappa) = \alpha_{R\to R}(\kappa^\bullet)$ for all κ, so the fitness of model $\Theta(\kappa^\bullet)$ in the unique linear equilibrium is

$$\mathbb{E}^\bullet \left[\mathbb{E}^\bullet \left[u_1^\bullet(\alpha^\bullet s_1, \alpha^\bullet s_2, \omega) \mid s_1\right]\right].$$

Given λ and entrant model $\Theta(\kappa)$, the entrant's fitness in the unique linear equilibrium is

$$\mathbb{E}^\bullet \left[\mathbb{E}^\bullet \left[(1-\lambda)u_1^\bullet(\alpha_{E\to R}(\kappa)s_1, \alpha_{R\to E}(\kappa)s_2, \omega) + (\lambda)u_1^\bullet(\alpha_{E\to E}(\kappa)s_1, \alpha_{E\to E}(\kappa)s_2, \omega) \mid s_1\right]\right].$$

Differentiate and evaluate at $\kappa = \kappa^\bullet$. At $\kappa = \kappa^\bullet$, agents with models Θ_R and Θ_E have the same fitness since they play the same strategies. So, a non-zero sign on the derivative would give the desired susceptibility to invasion from models with either slightly higher or slightly lower κ. This derivative is:

$$\mathbb{E}^\bullet \left[\mathbb{E}^\bullet \left[\begin{array}{c} \frac{\partial u_1^\bullet}{\partial q_1}(\alpha^\bullet s_1, \alpha^\bullet s_2, \omega) \cdot [(1-\lambda)\alpha'_{E\to R}(\kappa^\bullet) + \lambda\alpha'_{E\to E}(\kappa^\bullet)] \cdot s_1 \\ +\frac{\partial u_1^\bullet}{\partial q_2}(\alpha^\bullet s_1, \alpha^\bullet s_2, \omega) \cdot [(1-\lambda)\alpha'_{R\to E}(\kappa^\bullet) + \lambda\alpha'_{E\to E}(\kappa^\bullet)] \cdot s_2 \end{array} \middle| s_1 \right]\right].$$

Using the interim optimality part of Assumption SA1, $\mathbb{E}^\bullet \left[\frac{\partial u_1^\bullet}{\partial q_1}(\alpha^\bullet s_1, \alpha^\bullet s_2, \omega) \mid s_1\right] = 0$ for every $s_1 \in S$, using the necessity of the first-order condition. The derivative thus simplifies as claimed. ☐

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