

Learning from Viral Content*

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Abstract

We study learning on social media with an equilibrium model of users interacting with shared news stories. Rational users arrive sequentially, observe an original story (i.e., a private signal) and a sample of predecessors' stories in a news feed, and then decide which stories to share. The observed sample of stories depends on what predecessors share as well as the sampling algorithm generating news feeds. We focus on how often this algorithm selects more viral (i.e., widely shared) stories. Showing users viral stories can increase information aggregation, but it can also generate steady states where most shared stories are wrong. These misleading steady states self-perpetuate, as users who observe wrong stories develop wrong beliefs, and thus rationally continue to share them. Finally, we describe several consequences for platform design and robustness.

Keywords: social learning, selective equilibrium sharing, social media, platform design, endogenous virality

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1 Introduction

In recent years, viral content on social-media platforms has become a major source of news and information for many people. What content users consume often depends on the *news feeds* created by platforms like X, Facebook, and Reddit. Which stories go viral and which disappear is jointly determined by the algorithms generating these feeds and users' actions on the platforms (e.g., sharing, retweeting, or upvoting stories).

How does the design of the news feed affect how users learn on such platforms? Consider a platform deciding how much to push widely shared (or highly upvoted) content into users' news feeds. On the one hand, a news feed that primarily shows users widely shared stories can create a social version of the confirmation bias: incorrect but initially popular stories spread widely and determine people's beliefs, even though they are contradicted by most of the information that arrives later. One might expect such feedback loops with naive users, but we show they can also arise in an equilibrium model with rational users. The idea is that when stories supporting an incorrect position are shared more, subsequent users tend to see these incorrect stories in their news feeds due to the stories' popularity, and hence form incorrect beliefs through Bayesian updating. If users derive utility from sharing accurate content and thus share stories that agrees with their beliefs, they will rationally share these false stories and further increase their popularity. Users have less exposure to the true stories: even if these stories are more numerous, they are shared less than the false stories and therefore shown less by the news-feed algorithm.

But on the other hand, selecting news stories based on their popularity may help aggregate more information. Seeing a particular story in a news feed that selects widely shared content gives a user more information than the realization of a single signal. The popularity of this story also tells the user about the past sharing decisions of their predecessors, and thus lets the user draw inferences about the many stories that these predecessors saw in their news feeds. In some circumstances, seeing just a few stories in a news feed that emphasizes viral content can lead to strong Bayesian beliefs about the state of nature. This can happen even if individual stories are imprecise signals about the state, since sophisticated users can use the selection of these stories to infer much more about sharing on the platform.

This work examines the trade-offs in choosing how much to feature viral content in a news

feed and studies how this design choice affects social learning on the platform. There is an active public discussion about how news feeds shape society’s beliefs. Some commentators have blamed the widespread misinformation about issues ranging from public health to politics on social-media platforms pushing viral but inaccurate content into users’ feeds. We contribute to this discussion by developing an equilibrium model of people learning from news feeds and sharing news stories on a platform. We characterize learning outcomes under different news-feed designs, taking into account rational users’ responses to different designs and to other people’s equilibrium sharing patterns. The model also provides insights on specific applied questions, such as how news-feed designs can improve accuracy and when platforms are robust to manipulation by a malicious attacker.

In our model, a large number of users arrive in turn and learn about a binary state. Each user receives a conditionally independent binary signal about the state (which we call a *news story*) and observes a sample of stories from predecessors (which we call a *news feed*). These stories are sampled using a news-feed algorithm that interpolates between choosing a uniform sample of the past stories and choosing each story with probability proportional to its *popularity* (i.e., the number of times it has been shared). Users are Bayesians and know the news-feed algorithm, so they appropriately account for selection in the stories they see.¹ Users then choose which of these news-feed stories to share. We assume users prefer to share stories that match the true state, given their endogenous beliefs. This simple utility specification, which one might think is conducive to learning, can nevertheless generate rich learning dynamics including persistent learning failures.

News-feed algorithms in our model depend on a *virality weight* λ that captures the weight placed on popularity when generating news feeds: higher λ corresponds to showing more viral stories. The evolution of content on the platform is described by a stochastic process in $[0, 1]$ we call *viral accuracy*, which measures the relative popularity of the stories that match the true state in each period. We show viral accuracy almost surely converges to a (random) steady-state value, which depends on the randomness in signal realizations and in news-feed sampling. In equilibrium, there is always an *informative steady state* where most stories in news feeds match the state. But when the virality weight is high enough, there can also be a *misleading steady state* in equilibrium, where most stories in news feeds do not match the state (so viral accuracy is less than $\frac{1}{2}$). At a

¹An alternative approach would be to assume users are naive and fail to account for this selection. Many of the main forces we highlight in our equilibrium framework would also appear in this behavioral model.

misleading steady state, users tend to see false stories, and therefore believe in the wrong state and share these false stories. The misleading steady states correspond to the socially-generated confirmation bias described above.

These misleading steady states emerge when λ crosses a threshold, which we call the *critical virality weight* λ^* . Misleading steady states exist in equilibrium when virality weight is at or above this threshold, but not below it. A key finding is that this emergence is *discontinuous*: at the threshold virality level λ^* where the misleading steady state first appears, the probability of learning converging to this bad steady state is strictly positive. As a consequence, the accuracy of content on the platform jumps downward at this threshold. Below the critical virality weight, however, the unique informative steady state becomes monotonically more accurate as λ increases. This result formalizes the intuition mentioned above that a more viral news feed helps aggregate more information. Increasing λ therefore leads to a trade-off between facilitating more information aggregation and preventing the possibility of a misleading steady state in equilibrium.

Since misleading steady states only appear when virality weight exceeds the threshold λ^* , comparative statics of this threshold with respect to other parameters tell us which platform features make it more susceptible to misleading steady states. Platforms are more susceptible when news stories are not very precise, when news feeds are large, and when users share many stories. That is, misleading steady states arise on platforms that let users consume and interact with too much social information relative to the quality of their private information from other sources.

We give two consequences of our results with implications for the design and regulation of platforms. First, we describe a content-neutral policy that leads to better learning outcomes: letting the virality weight λ vary over time. Consider generating initial agents' news feeds with a low virality weight but later agents' news feeds with a high virality weight. We show there is a simple equilibrium that achieves high viral accuracy without producing misleading steady states. Intuitively, one way to improve learning is to let independent information accumulate early in the discussion of a given issue and then exploit the advantages of showing viral content later in the discussion. Second, we ask when a platform is robust to malicious attackers who manipulate its content. If a platform chooses λ sufficiently below the threshold λ^* , a large amount of manipulation is required to produce a misleading steady state. We provide a simple explicit lower bound on this amount, which we interpret as a robustness guarantee.

In our main model, agents observe the realizations of stories but not how viral those stories are. Motivated by existing social-media platforms, where users usually observe some information about the popularity of posts, we also consider a modified model where agents can distinguish between *viral* and *regular* stories in their news feeds. Each time a user shares a regular story, there is a chance of it “going viral” and creating a corresponding viral copy. A consistent conclusion that we find in both the main model and this modified model is that there must be misleading steady states when users see enough viral stories.

At a technical level, our paper applies mathematics techniques on stochastic approximation to an equilibrium model where agents respond optimally to the evolution of a stochastic process. The same techniques have been used in economics to study dynamics under behavioral heuristics (e.g., [Benaïm and Weibull \(2003\)](#) in evolutionary game theory or [Arieli, Babichenko, and Mueller-Frank \(2022\)](#) in naive social learning). Applying these tools to a setting where agents are best responding adds new challenges. Indeed, even for a fixed strategy, the system we study can often converge to multiple steady states and there is no closed-form expression for the probability of reaching a given steady state. Understanding outcomes under equilibrium sharing rules is even more complex. To make progress despite this complexity, we show that outcomes under a specific simple strategy (sharing stories that match a majority of one’s observations) tell us about the equilibrium outcomes (which cannot be characterized directly). In particular, a misleading steady state exists when users choose equilibrium sharing strategies if and only if one exists when users follow this simple strategy.

1.1 Related Literature

We first discuss how our model relates to a recent literature on learning from shared signals. Several papers have looked at different models of news sharing or signal sharing. As we discuss in detail below, the existing work focuses on the dissemination of a single signal, or on settings where signals are shared once with network neighbors but not subsequently re-shared. Our model differs on both of these dimensions. First, we consider a platform where many signals about the same state circulate simultaneously. These signals interact: a user’s social information consists of the multiple stories that they see in their news feed, so the probability that they share a given story depends on whether the other stories corroborate it or contradict it. Second, we allow signals to be shared widely through a central platform algorithm that generates news feeds for all users. A signal can

become popular due to early agents’ sharing decisions and get pushed into a later agent’s news feed, and this later agent can re-share the same signal. The combination of these two model features generates the social version of confirmation bias that we outlined earlier.

[Bowen, Dmitriev, and Galperti \(2023\)](#) study a model where signals are selectively shared at most once with network neighbors, but agents are misspecified and partially neglect this selection. This bias leads to mislearning, and it also generates polarization in social networks with echo chambers. By contrast, we focus on rational agents who make endogenous sharing decisions in equilibrium. [Bowen et al. \(2023\)](#) note that:

“[...]the Internet has also brought an abundance of information, which should lead people to learn quickly and beliefs to converge (not diverge) according to standard economic models.”

Our results imply that even if people observe a large (but finite) amount of information and rationally account for selection, they can converge to a misleading steady state if they mostly observe social information from peers (as may be the case on real-world social-media platforms). Indeed, our comparative statics results in [Section 3.5](#) show that the possibility of a misleading steady state arises precisely when users are exposed to more social information and less private information.

Another group of papers in operations research and economics study settings with “fake news” where people decide whether to share a story depending on the outcome of a (possibly noisy) fact check (e.g., [Papanastasiou \(2020\)](#), [Kranton and McAdams \(2023\)](#), and [Merlino, Pin, and Tabasso \(2023\)](#)) or depending on their prior beliefs about the story’s likelihood (e.g., [Bloch, Demange, and Kranton \(2018\)](#), [Acemoglu, Ozdaglar, and Siderius \(2022\)](#) and [Hsu, Ajorlou, and Jadbabaie \(2021\)](#)). Most of these papers consider the diffusion of a single signal that can be re-shared through a network, while [Kranton and McAdams \(2023\)](#) look at the supply-side decisions of information producers when consumers can share their stories at most once with network neighbors.² We focus on a different dimension of platform-design choices. Instead of asking about the network structure that connects users on the platform (e.g., echo chambers) or fact-checking technologies, we consider the platform’s choice in terms of showing its users more or less viral content.

[Buechel, Klößner, Meng, and Nassar \(2023\)](#), like our work, consider an environment where

²[Merlino, Pin, and Tabasso \(2023\)](#)’s model features one true and one false message.

agents can share and re-share copies of a signal. They study a model in which agents’ sharing behavior resembles the DeGroot heuristic. In particular, their agents’ sharing is independent of beliefs, while we study sharing rules that seek to share correct stories and therefore depend on beliefs.

Finally, there are some similarities between misleading steady states in our model and herding in observational social-learning models (Banerjee (1992), Bikhchandani, Hirshleifer, and Welch (1992), and a large subsequent literature³). In both cases, incorrect initial signals can lead to persistent wrong beliefs. A high-level distinction is that agents observe signals directly, rather than actions incorporating signals, but the observed signals are endogenously selected. At a more theoretical level, we consider an environment with an inherent constraint on the informativeness of social observations, where the expected accuracy of the news feeds is uniformly bounded away from 1 across all strategies of the agents. This leads to several new dynamics relative to the classical herding literature. First, agents learn imperfectly in the long run even without herding-type behavior, so we can quantitatively compare how long-run learning outcome changes across different platform parameters (e.g., Proposition 3). Such comparisons are key to the main trade-off between more information aggregation and misleading steady states. Second, misleading steady states persist in our model even though new private information continues to arrive and get shared on the platform. By contrast, the classical herding results rely sharply on the later agents’ private signals becoming completely lost once society reaches an information cascade.

2 Model

We consider a finite society with n people learning in sequence about an unknown state of nature $\omega \in \{-1, 1\}$. Everyone starts with the common prior that both states are equally likely. Each agent receives a binary private signal $s_i \in \{-1, 1\}$ about the state, interpreted as a *news story*. Call $s_i = -1$ a negative story and $s_i = 1$ a positive story. We assume stories are conditionally independent and symmetric, so that $\mathbb{P}[s_i = -1 | \omega = -1] = \mathbb{P}[s_i = 1 | \omega = 1] = q$ for some *story*

³Perhaps closest to our model within the observational social-learning literature, several papers assume agents observe a random sample of predecessors’ actions (Banerjee and Fudenberg (2004), Lévy, Pęski, and Vieille (2022), and Kabos and Meyer (2021)). Our techniques, meanwhile, are based on the same mathematics literature as Arieli, Babichenko, and Mueller-Frank (2022), who model the distribution of actions taken by agents as a generalized Pólya urn.

precision $0.5 < q < 1$. We also keep track of the *popularity score* of each story s_i , denoted $\rho(s_i)$. Each story starts with a score of 1 when it is first posted, and the score increases by 1 each time the story is shared by someone else.

We fix a *news feed* size $K \geq 1$. The first K agents receive no information other than their own news stories and mechanically post these stories onto the platform. (Our analysis would remain unchanged if we instead began with an exogenous finite pool of stories drawn from any distribution that is state-symmetric in the sense that the distribution over the number of stories matching the state does not depend on the state.) For each $i \geq K + 1$, agent i sees a news feed containing K stories posted by predecessors. The news feed only shows the realizations of the K sampled stories, but not their popularity scores or arrival times. Then, agent i *shares* C out of the K stories from their news feed, increasing each shared story's popularity score by 1, for some *capacity* $C \leq K/2$. Agent i gets utility $u > 0$ for each shared story that matches the state ω . Agent i also posts their own story s_i onto the platform.

The platform's *virality weight* $\lambda \in [0, 1]$ determines how it samples K stories to populate i 's news feed. For each of the K slots in the news feed, with probability λ , a story is sampled with probabilities proportional to the $i - 1$ stories' current popularity scores. With the complementary probability, a story is sampled uniformly at random from the $i - 1$ stories. We assume for simplicity that all stories are sampled with replacement (as we approach the steady state, the effect of replacement vanishes). All draws are independent.

The platform's sampling rule includes two special cases:

1. **Popularity-based sampling** ($\lambda = 1$): A story that has been shared twice as often as another has twice the probability of being sampled.
2. **Uniform sampling** ($\lambda = 0$): Predecessors' sharing decisions do not affect sampling.

More generally, sampling rules with λ between zero and one interpolate between these two cases. The virality weight λ measures how much the news feed shows more popular stories relative to random stories.

The n agents are uniformly randomly placed into the n positions, and do not know their positions. Each agent (correctly) believes that they are in each position $1, \dots, K$ with equal probabilities

if they do not observe a news feed, and in each position $K + 1, \dots, n$ with equal probabilities if they do observe a news feed. The informational environment is common knowledge.

2.1 Discussion and Interpretation

In this section, we discuss our interpretation of the model and the reasons behind our modeling choices. We begin by explaining connections to social-media platforms and then discuss our assumptions about users’ behavior and information.

Our Model of Platforms. We view the news stories s_i as original content that users discover from external sources (e.g., opinion pieces from local newspapers, new scientific preprints, etc.) and post on the social-media platform (e.g., X, Facebook, or Reddit). We assume that agents always post their stories to ensure that new private information continues to arrive and spread on the platform, but could easily adapt our results to other assumptions about the information arrival process.

The platform presents each user with a news feed of stories that others have posted, and gives users some way of expressing endorsement for the content discovered by others. What we generically refer to as “sharing” in our model corresponds to retweeting on X, reposting content you saw on Facebook on your own timeline, upvoting a story on Reddit, and so forth.

The news-feed algorithm determines what content is shown. It can focus on showing more viral content (larger λ) or more “random” content (smaller λ). Displaying “random” content could represent, for example, showing a user the most recent stories that their friends posted without regard for the stories’ popularity score. The functional form we chose above to model this trade-off has the particularly useful property that the total popularity scores of positive and negative stories are a sufficient statistic for the distribution of sampled stories, but other functional forms are also possible. In particular, platforms could also use more extreme sampling rules that place more weight on more viral stories than popularity-based sampling (e.g., the probability of sampling a story is proportional to the square of its popularity).

The virality of the news feed is a design choice that social-media companies devote substantial attention to in practice. Over the years, different iterations of the X (Twitter) feed gave different levels of emphasis on the trending or most popular tweets on the platform. For Reddit, its ordering algorithm for displaying posts on the front page evolved over a decade. An entry on Reddit’s

company blog in 2009 shows the platform designers are well aware of the disadvantages of putting too much emphasis on the most highly upvoted content:

“Once a comment gets a few early upvotes, it’s moved to the top. The higher something is listed, the more likely it is to be read (and voted on), and the more votes the comment gets. It’s a feedback loop” (Munroe, 2009).

User Behavior. Turning to user behavior, we assume that users are rational and want to share stories that match the true state. We will see that even under these assumptions, which we view as relatively conducive to learning, there are often misleading steady states. The assumption that users want to share correct content is also motivated by empirical evidence on sharing behavior. In laboratory experiments, Pennycook et al. (2020, 2021) find people have an intrinsic preference for sharing news from more trustworthy sources, which are more likely to accurately reflect the state. We will see in Section 3.3 that our analysis is robust to including other motivations for sharing if users also care enough about sharing accurate content. But different types of preferences could lead to different dynamics. In particular, if users care primarily about influencing the long-run accuracy of content on the platform or an eventual societal decision, then it might be possible for them to avoid misleading steady states.

We also consider a symmetric prior over the two states, which allows us to consider a restricted class of strategies and simplifies best responses in the absence of misleading steady states. Exact symmetry of priors is not needed for our analysis (see Section 3.3), but dropping our symmetry assumption on strategies would considerably complicate the analysis.

We assume an explicit capacity constraint C on how much people can share. The primary reason is tractability, as the analysis is considerably cleaner when the number of stories shared does not depend on the realizations of the sampled stories. A capacity constraint also captures the fact that people tend to only share a small fraction of the content that they consume on social-media platforms.⁴ Note that even if the agents are not forced to share exactly C stories out of the K in their news feeds, they would still find it optimal to share up to the capacity constraint. This is because they incur no marginal cost from sharing and no penalty from sharing an incorrect story,

⁴For example, even among the top 10% of tweeters on X, the median number of favorites per month is 70 and the median number of tweets per month is 138, which are presumably much lower than the number of tweets that these users read per month (Wojcik and Hughes, 2019).

and they will always believe that each story has a positive probability of being correct.

In our main model, agents do not see the current popularity scores or the arrival times of the stories in their news feeds. This assumption is motivated by the difficulty of inferring the state from the popularity or age of observed posts. Indeed, in reality these inferences are likely complicated further by characteristics of news stories outside of our model.⁵ In Section 5, we allow agents to observe some information about the popularity of stories. We also assume that people do not know their order in the sequence. This is likely more realistic than assuming that everyone knows their precise order. From a technical perspective, it is also the more tractable assumption that lets us focus on analyzing long-run steady states.⁶

2.2 Strategy, Symmetric BNE in Finite Societies, and Limit Equilibrium

An agent who does not see a news feed has no decisions to make. We therefore define a mixed strategy in the game to be $\sigma : \{-1, 1\} \times \{0, \dots, K\} \rightarrow \Delta(\{0, 1, \dots, C\})$, so that $\sigma(s, k)$ gives the distribution over the number of positive stories shared when the agent discovers the story s and sees a news feed with k positive stories out of K .⁷ We will regard the space of mixed strategies as a subset of $\mathbb{R}^{2(K+1)(C+1)}$ with the standard Euclidean norm. Mixed strategies must satisfy feasibility constraints in terms of the available numbers of positive and negative stories to share, so $\sigma(s, k)$ cannot have values larger than k or smaller than $C + k - K$ in its support for any $0 \leq k \leq K$. Note that we only need to discuss positive stories since the agent must always share C stories in total.

A simple strategy, which will play a central role in our analysis, is to follow the majority of the stories in the news feed (breaking ties in favor of the private signal):

Definition 1. The *majority rule* is the strategy defined by $\sigma^{\text{maj}}(s, k)(C) = 1$ if either $k > K/2$ or $k = K/2$ and $s = 1$, and $\sigma^{\text{maj}}(s, k)(0) = 1$ otherwise.

The majority rule is a pure strategy that either shares C stories with the realization of 1 or C stories with the realization of -1 , depending on the news-feed majority. Note the majority rule is

⁵Vosoughi, Roy, and Aral (2018) find that fake news spreads more widely than accurate stories, and Altay, De Araujo, and Mercier (2022) argue this is because fake news stories are more interesting to users.

⁶In our model where agents hold a uniform prior over positions, we will be able to analyze changes in the platform over time without needing to account for time-varying strategies. If agents know their positions, strategies and the content on the platform are both changing over time, and even basic convergence properties are unclear.

⁷Agents cannot distinguish between different positive (or negative) stories in their news feeds. Moreover, which positive (or negative) stories they share does not affect subsequent agents' observations under the family of sampling rules we consider.

feasible because we have assumed $C \leq K/2$. The majority rule need not be an equilibrium strategy in general — intuitively, it is only optimal when stories in agents’ news feeds are more informative than their private stories. Nevertheless, it turns out the majority rule will help us understand the qualitative properties of equilibrium outcomes even when it is not itself an equilibrium.

We apply the solution concept of Bayesian Nash equilibrium (BNE). Note that all possible observations are on-path given any strategy profile. We focus on player-symmetric and state-symmetric BNE: that is, a BNE where each agent uses the same strategy σ , and σ treats positive and negative stories symmetrically.⁸ We abbreviate this refinement as “symmetric BNE.”

We are mainly interested in analyzing the limits of symmetric BNE when the number of agents on the platform grows large, and studying the accuracy of the resulting news feeds in the long run. Such a limit is well defined because for fixed parameters q, K, C, λ , the space of strategies stays constant as the number of agents n grows.

Definition 2. For fixed q, K, C, λ parameters, a mixed strategy σ^* is a *limit equilibrium* if there exists a sequence of symmetric BNE $(\sigma^{(j)})_{j=1}^\infty$ for finite societies with n_j agents and the same q, K, C, λ parameters, where $n_j \rightarrow \infty$ and $\lim_{j \rightarrow \infty} \sigma^{(j)} = \sigma^*$.

Symmetric BNE and limit equilibria both exist for all parameter values:

Proposition 1. *For any finite n and parameters q, K, C, λ , there exists a symmetric BNE. For any parameters q, K, C, λ , there exists a limit equilibrium.*

The proof shows that there is a symmetric BNE for all n via a standard fixed-point argument. Since the space of feasible mixed strategies can be viewed as a compact subset of a finite-dimensional Euclidean space, a limit equilibrium must exist.

3 Steady States and Equilibrium Steady States

We begin this section by describing the structure of steady states under a fixed strategy σ . We then distinguish informative and misleading steady states. The third subsection describes equilibrium strategies and the structure of steady states under equilibrium behavior. We then provide numerical

⁸More precisely, state symmetry means that for every $s \in \{-1, 1\}$ and $0 \leq k \leq K$, we have $\sigma(s, k)(z) = \sigma(-s, K - k)(C - z)$ for each $0 \leq z \leq C$.

illustrations of equilibria and the corresponding steady states. The final section considers how the structure of equilibrium steady states changes as we change the model parameters.

3.1 Definition and Characterization of Steady States

Suppose everyone uses the same strategy σ , which needs not be an equilibrium, and the true state is ω . How will the total popularity score of the correct stories that match the state compare with that of the incorrect stories in the long run? We define the concept of steady states to study this question.

Given the true state of nature ω , a finite set of t stories (s_1, \dots, s_t) on the platform and the popularity scores of these stories $\rho(s_i)$, the *viral accuracy* of the platform is defined to be

$$x(t) = \frac{\sum_{i:s_i=\omega} \rho(s_i)}{\sum_{j=1}^t \rho(s_j)}.$$

Viral accuracy measures the relative popularity of the stories that match the true state. Imagine a society with infinitely many agents, with all agents $i \geq K + 1$ using the strategy σ . This induces a stochastic process $(x(t))_{t=1}^\infty$ where $x(t)$ is the viral accuracy of the platform after agent t has acted.

Definition 3. A point x^* such that $x(t) \rightarrow x^*$ with positive probability is a *steady state* of the strategy σ .

When viral accuracy converges to a steady state x^* , roughly x^* fraction of the total popularity score on the platform is associated with correct stories in all late enough periods. This fraction persists as fresh stories get posted on the platform each period and agents use σ to decide which stories to share from their randomly generated news feeds. The next result tells us that for any state-symmetric strategy, viral accuracy almost surely converges, and the set of steady states X^* is finite.

Proposition 2. *Given a state-symmetric strategy σ , there is a finite set of steady states $X^* \subseteq (0, 1)$ such that when all agents use σ , almost surely $x(t) \rightarrow x^*$ for some $x^* \in X^*$.*

The proof uses a convergence result from stochastic approximation (Theorem 2.1 from Chapter 2 of [Borkar, 2023](#)). When X^* contains at least two elements, the limit steady state $x^* \in X^*$ is

random and can depend on the early agents' private signal realizations and the random sampling involved in generating news feeds.

The basic idea behind the proof of Proposition 2, as well as much of the subsequent analysis, is that we can describe the state of the system in terms of $x(t)$ and the fraction of stories up to time t that match the state. We can decompose the change $x(t+1) - x(t)$ between periods into a deterministic term (which we will approximate using the inflow accuracy function defined below) and a martingale noise term. Given this decomposition, a martingale convergence theorem shows that $x(t)$ converges almost surely.

In light of Proposition 2, we write $\pi(\cdot \mid \sigma)$ for the distribution over steady states generated by a state-symmetric strategy σ .⁹ A substantial challenge in analyzing our model is that we cannot obtain closed-form expressions for $\pi(\cdot \mid \sigma)$ as these probabilities depend on a complicated stochastic process. We focus instead on understanding when the support of $\pi(\cdot \mid \sigma)$, which is the set of steady states, contains certain values of x . We will see this question is already highly non-trivial, and the answers will have interesting implications for understanding equilibrium and design questions.

Our next result will characterize the support of the distribution $\pi(\cdot \mid \sigma)$ over steady states in terms of the fixed points of an *inflow accuracy function*, which we now define. Suppose today's viral accuracy is x , and exactly q fraction of the stories on the platform are correct. A new agent must increase the total popularity score on the platform by $C + 1$, as they share C existing stories and post a new story. We define the inflow accuracy function $\phi_\sigma(x)$ to be the expected fraction of the incoming $C + 1$ popularity score that will be allocated to stories that match the state.

Definition 4. The *inflow accuracy function* is

$$\phi_\sigma(x) := \frac{q + \sum_{k=0}^K P_k(x, \lambda) \cdot [q \cdot \mathbb{E}[\sigma(1, k)] + (1 - q) \cdot \mathbb{E}[\sigma(-1, k)]]}{1 + C}$$

where $P_k(x, \lambda) := \mathbb{P}[\text{Binom}(K, \lambda x + (1 - \lambda)q) = k]$ and $\text{Binom}(K, p)$ is the binomial distribution with K trials and success probability p .

To understand the formula in the definition, note that $\lambda x + (1 - \lambda)q$ is the *sampling accuracy* of the platform: the probability of each news-feed story being correct, given viral accuracy x and

⁹By symmetry of σ and of the environment, the distribution over steady states is the same conditioning on $\omega = 1$ and $\omega = -1$.

virality weight λ . We can use sampling accuracy to express the probability of getting k positive stories out of K in the news feed when $\omega = 1$ for every $0 \leq k \leq K$, then consider how the strategy σ combines the private signal s_i and the number of positive stories in the news feed to make a sharing decision. Finally, $\phi_\sigma(x)$ also takes into account that the story posted by the agent, which starts with a popularity score of 1, has q chance of matching the state. While $\phi_\sigma(x)$ is defined in terms of the expected fraction of the new popularity score assigned to correct stories when $\omega = 1$, the symmetry of the environment and of σ implies that it also describes the same fraction when $\omega = -1$.

We always have $\phi_\sigma(0) > 0$ and $\phi_\sigma(1) < 1$. The idea is that if $x \approx 0$ and almost all of the popularity score are associated with the wrong stories, then the arrival of fresh stories posted by new agents tends to increase x , as a majority of these stories match the state. If on the other hand $x \approx 1$, then these fresh stories will on average lower x , since they have a non-zero probability of mismatching the state. So ϕ_σ must have a fixed point by continuity.

A fixed point of the inflow accuracy function ϕ_σ is a natural candidate for a steady state induced by σ , as it intuitively represents a level of viral accuracy that tends to be exactly maintained on average by the inflow of new popularity score, on a platform with sufficiently many stories so that approximately q fraction of them match the true state. The next result establishes this formally, provided the fixed point is not unstable from both sides.

Theorem 1. *We have $\pi(x^* \mid \sigma) > 0$ if $\phi_\sigma(x^*) = x^*$ and there exists some $\epsilon > 0$ so that either (a) $\phi_\sigma(x) < x$ for all $x \in (x^*, x^* + \epsilon)$, or (b) $\phi_\sigma(x) > x$ for all $x \in (x^* - \epsilon, x^*)$. Conversely, for $x^* \in [0, 1]$, we have $\pi(x^* \mid \sigma) > 0$ only if $\phi_\sigma(x^*) = x^*$.*

We first discuss fixed points which are stable from both sides. As an example, Figure 1 plots the inflow accuracy function for the majority rule when $K = 7$, $C = 3$, $q = 0.55$, and $\lambda = 1$. There are two fixed points that are stable from both sides, and Theorem 1 implies both are steady states. At the upper fixed point, stories matching the state are more popular. At the lower fixed point, however, incorrect stories are more popular than correct stories. Under the majority rule, such a misleading state is reached with positive probability: if enough initial stories are incorrect, the majority rule will continue sharing incorrect stories. We will see in Section 3.3 these misleading steady states can also arise under equilibrium behavior.

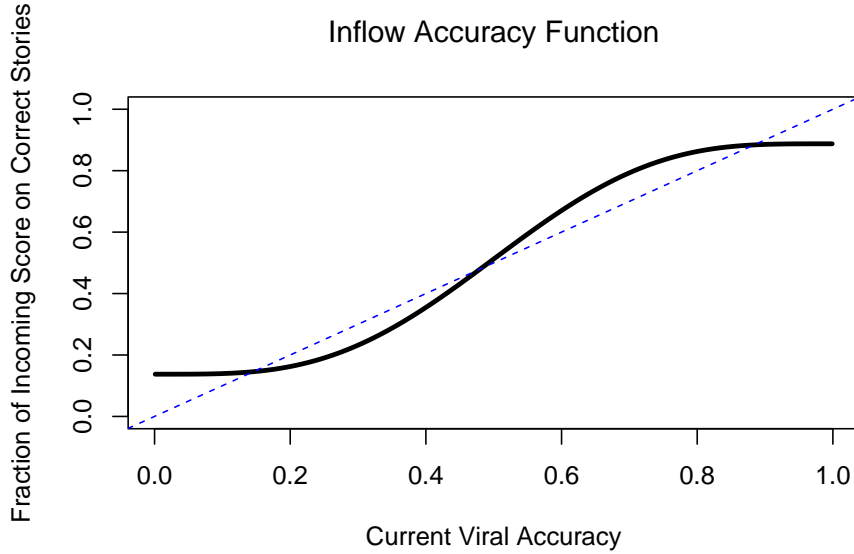


Figure 1: The inflow accuracy function for the majority rule with $K = 7$, $C = 3$, $q = 0.55$, $\lambda = 1$.

The more subtle case is a fixed point of ϕ_σ that is unstable from one side (see Figure 2 for an illustration). A *touchpoint* of ϕ_σ is a fixed point $x^* = \phi_\sigma(x^*)$ where exactly one of condition (a) or condition (b) from Theorem 1 holds (so x^* is only stable from one side). Theorem 1 says that if ϕ_σ has a touchpoint x^* , then viral accuracy converges to x^* with positive probability. This means the distribution over steady states is discontinuous in the strategies that agents use and discontinuous in parameters of the model such as λ and q , and we will discuss some consequences of the discontinuity below.

One might expect that the stochastic process of viral accuracy should not converge with positive probability to fixed points of ϕ_σ which are unstable from one side, because random noise in the process $x(t)$ can bring it to the unstable side of the fixed point. But careful analysis shows that there is a positive probability event that $x(t)$ converges to the touchpoint x^* while entirely staying on the stable side, with the noise terms never being large enough to move the process over to the unstable side. This is because the noise terms from a single agent's sharing choices become smaller over time and do so sufficiently quickly. The proof extends the techniques of Pemantle (1991), which shows a similar result for generalized Pólya urns.¹⁰

¹⁰Our model does not fit Pemantle (1991)'s definition of a generalized Pólya urn because (1) the relevant stochastic process in our model is two-dimensional since we keep track of both the fraction of stories that match the true state and the viral accuracy, and (2) signals are shared in correlated groups of $C + 1$ signals rather than one at a time.

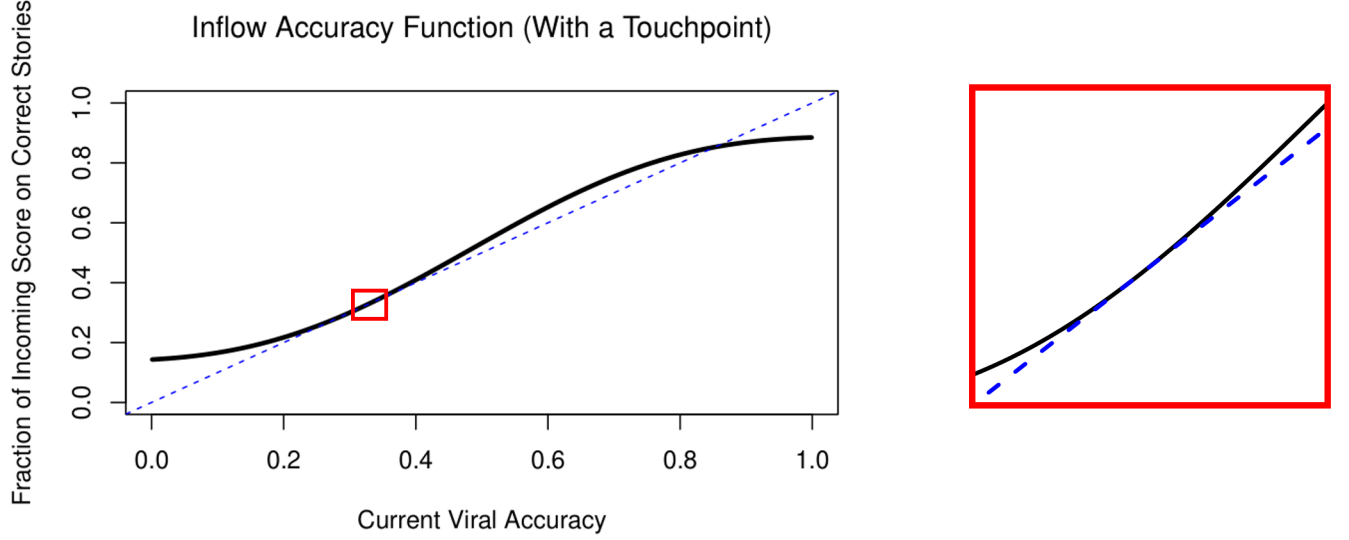


Figure 2: The inflow accuracy function for the majority rule with $K = 7$, $C = 3$, $q = 0.55$, $\lambda \approx 0.76$. Here $\phi_{\sigma^{\text{maj}}}$ has two fixed points: the left fixed point is a touchpoint that is only stable from the left side (the red box shows a zoomed-in view). The right fixed point is stable from both sides. Theorem 1 implies viral accuracy has a positive probability of converging to each of these two fixed points.

3.2 Informative and Misleading Steady States

We may classify steady states into two types. One type is an informative steady state, where sampling accuracy is above $1/2$ and it is more likely that stories in the news feed are true. The other type is a misleading steady state, where the opposite happens.

Definition 5. A steady state x is *informative* if $\lambda x + (1 - \lambda)q \geq 1/2$, and *strictly informative* if this inequality is strict. A steady state x is *misleading* if $\lambda x + (1 - \lambda)q \leq 1/2$, and *strictly misleading* if this inequality is strict.

Even reasonable strategies like the majority rule σ^{maj} can generate misleading steady states. Recall from Figure 1 and the discussion after Theorem 1 that σ^{maj} has two steady states with the parameters $K = 7$, $C = 3$, $q = 0.55$, and $\lambda = 1$. One is informative, but the other is misleading.

In a misleading steady state, the virality of false stories becomes self-sustaining. The state might be $\omega = 1$ but most people see negative stories in their news feeds, as the platform's virality weight implies the popular false stories tend to get shown to users. This happens even though the total number of negative stories is smaller than the total number of positive stories on the platform.

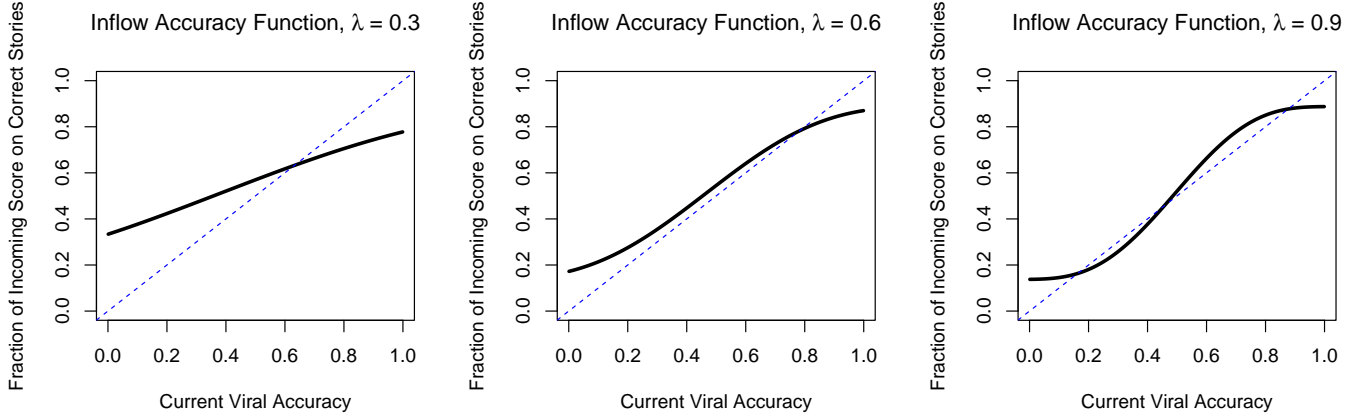


Figure 3: The inflow accuracy function for the majority rule with $q = 0.55$, $K = 7$, $C = 3$, and $\lambda \in \{0.3, 0.6, 0.9\}$. With $\lambda = 0.3$ and $\lambda = 0.6$, there is a single informative steady state. With $\lambda = 0.9$, a misleading steady state appears.

Under the majority rule σ^{maj} , for example, agents will then share the negative stories from their news feeds, which further perpetuates these stories' popularity and makes them more likely to be seen by future agents.

We will see that $\phi_{\sigma^{\text{maj}}}$, the inflow accuracy function associated with the majority rule, plays an important role in determining the equilibrium steady states of *any* limit equilibrium. As a first step in this direction, we observe that the steady states of the majority rule σ^{maj} satisfy the following useful properties:

Lemma 1. *If x is a steady state of σ^{maj} , then it is strictly informative if and only if $x > 1/2$, and strictly misleading if and only if $x < 1/2$. Also, $x = 1/2$ is not a fixed point of $\phi_{\sigma^{\text{maj}}}$.*

Steady states are generally classified as informative or misleading depending on the sampling accuracy. The lemma says for the majority rule, we can equivalently classify steady states based on whether viral accuracy is larger than $1/2$.

The number of steady states for a fixed strategy depends on λ . The three figures below plot the inflow accuracy function with $q = 0.55$, $K = 7$, $C = 3$ and the majority rule. The three plots in Figure 3 correspond to three different values of λ : $\lambda = 0.3$, $\lambda = 0.6$, and $\lambda = 0.9$. When $\lambda = 0.3$ and $\lambda = 0.6$, there is only an informative steady state, and this steady state is more accurate when $\lambda = 0.6$. But when $\lambda = 0.9$, there is both an informative steady state and a misleading steady state.

3.3 Equilibrium Steady States

So far, we have discussed steady states associated with arbitrary strategies. We are mainly interested in *equilibrium steady states*, that is the distribution $\pi(\cdot \mid \sigma^*)$ when σ^* is a limit equilibrium strategy.

We now define the critical virality weight, which is the smallest λ for which there is a misleading steady state under the majority rule. We will see the set of equilibrium steady states changes sharply around this critical value of λ .

Definition 6. The *critical virality weight* λ^* is

$$\lambda^* := \inf\{\lambda \in [0, 1] : \phi_{\sigma^{\text{maj}}}(x^*) = x^* \text{ for some } x^* \in [0, 1/2]\},$$

provided this set is non-empty. Otherwise, we let $\lambda^* = \infty$.

Depending on the values of the parameters q, K, C , it turns out that either σ^{maj} only has strictly informative steady states for any virality weight (so $\lambda^* = \infty$), or there is some smallest $0 < \lambda^* \leq 1$ where a fixed point in $[0, 1/2]$ first appears for $\phi_{\sigma^{\text{maj}}}$. For instance, for $q = 0.55, K = 7, C = 3$, Figure 2 shows that $\lambda^* \approx 0.76$. The next theorem fully characterizes when misleading steady states exist across all limit equilibria for every level of λ .

Theorem 2. For $0 < \lambda \leq \lambda^*$, the unique limit equilibrium is σ^{maj} . At every $\lambda < \lambda^*$, σ^{maj} only has one equilibrium steady state, and it is strictly higher than q (thus, strictly informative). For $\lambda \geq \lambda^*$, every limit equilibrium induces at least one strictly misleading steady state.

This result shows how the platform's virality weight affects the types of equilibrium steady states: there are only informative equilibrium steady states when $\lambda < \lambda^*$, while there will always be misleading equilibrium steady states when $\lambda \geq \lambda^*$.¹¹ It also shows the majority rule is the only possible limit equilibrium for non-zero virality weights below the critical virality weight λ^* .¹² For virality weights above λ^* , the majority rule may not be a limit equilibrium, and there may be multiple limit equilibria. Nevertheless, the result tells us that every limit equilibrium has a positive

¹¹We will see in Proposition 5 that λ^* is finite whenever K and C are large enough, so misleading steady states can indeed arise.

¹²When $\lambda = 0$, the only possible equilibrium steady state is q , so every story in the news feed is exactly as informative as one's private signal. There is thus some degree of freedom in tie-breaking when there is one more positive story than negative story in the news feed and one's private signal is negative.

probability of generating a misleading steady state where false stories dominate news feeds. Users are aware of the possibility of a misleading steady state and would like to correct for it, but are unsure whether society converged to a misleading steady state or an informative one.

The theorem greatly simplifies checking whether there is a misleading steady state at a limit equilibrium under given parameter values. Without the theorem, checking for misleading steady states would require solving for equilibrium strategies, which is a complicated calculation depending on $\pi(\cdot|\sigma)$ and therefore the entire stochastic process. The theorem says we can instead check for misleading steady states under the majority rule, which are simply roots of a polynomial.

This reduction relies on two properties, established in the proof: (1) if any strategy sustains a misleading steady state, then the majority rule does too,¹³ and (2) when there are no misleading steady states, the majority rule is a best response if the number of agents n in the society is sufficiently large. These properties give an intuition for why rational agents fail to prevent misleading steady states. For parameter values where the majority rule sustains a misleading steady state, there are of course other symmetric strategy profiles that only admit informative steady states. But no such strategy can be a limit equilibrium, because rational agents prefer to deviate to the majority rule in the absence of misleading steady states. So *every* limit equilibrium must sustain misleading states if the majority rule does so.

Property (2) is robust to some variation in agents' beliefs and preferences. Majority rule is a strict equilibrium whenever there is no misleading steady state, and can give a substantially higher payoff than other strategies in some cases (e.g., when the news feed size K is even and signals are precise). So steady states under equilibrium strategies can have the same basic structure if agents want to share accurate stories but also have other motivations (e.g., sharing stories that will be shared by others or influencing others' beliefs). Similarly we can allow asymmetric priors over the two states. Naturally very different preferences or priors could lead to different dynamics.

Finally, Theorem 1 and Theorem 2 together imply a discontinuity in equilibrium learning outcomes at λ^* . For λ just below λ^* , we converge almost surely to a steady state where a majority of users believe the true state is more likely. But when $\lambda = \lambda^*$, there is a positive probability of converging to a misleading steady state. The expected accuracy under the limit equilibrium

¹³This makes use of the capacity constraint model of sharing. If the number of stories shared depended on the realization of the sampled signals, it seems plausible there could be a misleading steady state under a strategy that sometimes shares fewer stories than the majority rule but not under the majority rule.

strategy σ^{maj} also discontinuously drops at λ^* :

Corollary 1. *The expected steady-state viral accuracy under the unique limit equilibrium jumps downward at λ^* :*

$$\lim_{\lambda \rightarrow (\lambda^*)^-} \mathbb{E}_{\pi(x^* | \sigma^{\text{maj}}, \lambda)}[x^*] > \mathbb{E}_{\pi(x^* | \sigma^{\text{maj}}, \lambda^*)}[x^*].$$

The corollary means that outcomes can be very sensitive to design choices, and we discuss one consequence in Section 4.2.

We now turn to benefits of higher virality weight λ^* . Our next result says that larger a virality weight can lead to a stronger consensus:

Proposition 3. *For $\lambda \in [0, \lambda^*)$, the unique steady state x^* at the unique limit equilibrium under λ is strictly increasing in λ .*

In the region of virality weights that do not generate misleading equilibrium steady states, increasing λ allows more information aggregation. This is because a positive story in i 's news feed not only tells i about the realization of a single signal, but also lets i draw inferences about the hidden information available to i 's predecessors who may have chosen to share that positive story. As λ increases, the sampled news-feed stories are closer to indicating a consensus among many agents.

We state the proposition for $\lambda < \lambda^*$, but a similar argument also shows that increasing λ tends to increase consensus in other regions as well. Formally, consider any interval $(\underline{\lambda}, \bar{\lambda})$ on which the set of limit equilibria and the set of steady states under each equilibrium are unchanged. Fix a limit equilibrium σ^* and a continuous selection $x^*(\lambda)$ of a steady state for each $\lambda \in (\underline{\lambda}, \bar{\lambda})$. Then $|x^*(\lambda) - \frac{1}{2}|$ is strictly increasing in λ on the interval $(\underline{\lambda}, \bar{\lambda})$. Informative steady states become more accurate, while misleading steady states induce a stronger wrong consensus.

Theorem 2 and Proposition 3 together formalize the trade-off in the virality weight λ described in the introduction. Increasing λ initially increases the steady-state viral accuracy and sampling accuracy. But starting at a critical threshold λ^* , it discontinuously creates the social form of confirmation bias discussed in the introduction, which we have now formalized in terms of a misleading steady state.

3.4 Numerical Illustrations of Equilibrium for $\lambda > \lambda^*$

Theorem 2 determines the unique equilibrium for $\lambda \leq \lambda^*$ but does not give a complete characterization when $\lambda > \lambda^*$. To illustrate equilibrium behavior and the distribution of long-run viral accuracy on platforms in this region, we describe several numerical examples. We find agents adjust their behavior considerably in response to the possibility of misleading steady states, but nevertheless converge to misleading steady states quite often.

We numerically estimated equilibria for news feed sizes $K \in \{6, 8, 10\}$ in an example with virality weight $\lambda = 1$, story precision $q = 0.55$, and sharing capacity $C = 3$. The virality weight $\lambda = 1$ is strictly higher than the critical virality weight λ^* in each of these environments. For each K , we first check for a pure strategy equilibrium and then check for a mixed strategy equilibrium of a particular form (see details in Appendix B). In all cases this procedure finds a single limit equilibrium, which we describe below.

We find that the limit equilibrium is mixed for $K = 6$. If at least three out of the six stories in the news feed match the agent's private signal s_i , then the agent always shares three news-feed stories that match s_i . If two out of the six news-feed stories match s_i , then the agent shares the two stories matching s_i (and one other story) with probability 19.26%, and with the complementary probability shares three stories that do not match s_i . If fewer than two news-feed stories match s_i , then the agent shares three stories that do not match s_i .

Similarly, the limit equilibrium is also mixed for $K = 8$. If at least four out of the eight stories in the news feed match the agent's private signal s_i , then the agent always shares three news-feed stories that match s_i . If three out of the eight news-feed stories match s_i , then the agent shares the three stories matching s_i with probability 72.11%, and with the complementary probability shares three stories that do not match s_i . If fewer than three news-feed stories match s_i , then the agent shares three stories that do not match s_i .

We find a different equilibrium structure for $K = 10$: the limit equilibrium is a pure strategy. If there is a supermajority of at least seven stories with the same realization in the news feed, then the agent shares three news-feed stories from the majority side. Otherwise, the agent shares three stories that match their private signal s_i .

The intuition behind these equilibria is that the possibility of the platform being stuck in a

misleading steady state makes news-feed stories less informative about the state of the world. When $\lambda < \lambda^*$, news-feed stories are strictly more informative than private signals. In these equilibria, however, an observation of k positive stories and $k - 2$ negative stories in the news feed is either exactly as informative as one positive private signal (as in the mixed equilibria for $K = 6$ and $K = 8$) or strictly less informative (as in the pure equilibrium for $K = 10$).

Under the estimated limit equilibria, we simulate the evolution of content on the platform in 10,000 trials with 40,000 periods each. We find that the viral accuracy is below $\frac{1}{2}$ after 40,000 periods 35.7% of the time with $K = 6$, 28.2% of the time with $K = 8$, and 28.1% of the time with $K = 10$. The probability of viral accuracy being near the misleading steady state after a large number of periods is therefore quite substantial in each case. This suggests that society often converges to misleading steady states, even when users adjust their behavior to follow private signals more often.

3.5 Comparative Statics

Theorem 2 characterizes the critical virality weight λ^* as the smallest value of λ such that the polynomial $\phi_{\sigma^{\text{maj}}}(x)$ has a touchpoint. We now use this characterization to show how the critical virality weight changes with respect to parameters of the environment. A lower critical virality weight means an equilibrium misleading steady state exists for a larger set of sampling rules. So we interpret the exercise as asking what properties of the environment and social-media platform lead to situations where false stories generate news feeds, holding fixed the platform's sampling rule.

Proposition 4. *Let $\lambda^*(q, K, C)$ be the critical virality weight for parameters (q, K, C) .*

- $\lambda^*(q', K, C) \geq \lambda^*(q, K, C)$ if $q' > q$
- $\lambda^*(q, K, C') \geq \lambda^*(q, K, C)$ if $C' < C$
- $\lambda^*(q, K - 2, C) \geq \lambda^*(q, K, C)$ for every K
- $\lambda^*(q, K + 1, C) \geq \lambda^*(q, K, C)$ if K is odd
- $\lambda^*(q, K - 1, C) \geq \lambda^*(q, K, C)$ if K is odd

All of these inequalities are strict whenever $\lambda^(q, K, C)$ is finite.*

The overall message is that misleading steady states emerge when individuals consume and interact with a large amount of social information on the platform relative to the amount of private information that they have from other sources. It is easier to generate a misleading equilibrium steady state when individual stories are noisy signals of the state of nature (so there is little private information), when users can share a large number of stories in their news feeds, and when users read many stories on the social-media platform. Intuitively, since each individual's private information is untainted by the popularity of various stories on the social-media platform, a sufficient amount of high-quality private information can counteract the harmful effects of trending false stories in news feeds. On the other hand, if users spend a great deal of time on the platform, browsing many stories in their news feeds and sharing many of the stories they read, then they are more likely to fall prey to a misleading steady state and help perpetuate the virality of inaccurate news.

The comparative statics in K depend on the parity of K . Misleading steady states are less likely for even K because agents with symmetric social observations (i.e., $K/2$ news-feed stories matching each state) follow their private signals, which match the true state more frequently. Aside from this detail, however, increasing K does create misleading steady states for a larger set of parameters.

While Proposition 4 tells us the direction in which λ^* changes as K and C increase, we may still want to know the extent to which these two parameters can affect the critical virality weight. The next result shows the full range of possible critical virality weight values when we fix the story precision and change how many news stories people read and share.

Proposition 5. *For any q, K, C , we have $\lambda^*(q, K, C) > 1 - \frac{1}{2q}$. But for any fixed $1/2 < q < 1$ and any $\bar{\lambda} > 1 - \frac{1}{2q}$, there exist $\underline{K}, \underline{C}$ so that whenever $K \geq \underline{K}$, $C \geq \underline{C}$, we have $\lambda^*(q, K, C) \leq \bar{\lambda}$.*

Proposition 5 says that if users have large enough news feeds and share sufficiently many stories from their feeds, then the critical virality weight will be arbitrarily close to $1 - \frac{1}{2q}$. In particular, no matter how precise the individual stories, every limit equilibrium admits a misleading steady state when $\lambda \geq 1/2$ provided K and C are large enough. But $1 - \frac{1}{2q}$ is also a sharp lower bound on the critical virality weight, so news-feed algorithms close enough to uniform sampling remain immune to the possibility of misleading steady states even if we let users access and interact with more and more social information.

4 Platform Design

In this section, we provide several implications of our results for platform design. We first show that if the platform can change the virality weight λ over time, then a simple policy can guarantee a high viral accuracy in large societies. We then discuss when social learning outcomes on a platform are robust to exogenous manipulation.

4.1 Changing Virality Weight Over Time

Our main model analyzed platform dynamics with a fixed virality weight and found a basic trade-off: higher virality weights can improve the accuracy of informative steady states but can also lead to misleading steady states. A natural question is whether sampling rules outside the class we have considered can circumvent this trade-off and improve accuracy. We now show that a simple modification can do so: letting the virality weight vary over time. This corresponds to generating news feeds with different algorithms when an issue first emerges and after the discussion has developed further.

To formalize this, let \bar{x} be the viral accuracy at the informative steady state under the majority rule when $\lambda = 1$. We will show that \bar{x} is the best possible steady state viral accuracy that can be reached with positive probability given any \bar{x} and any *fixed* virality weight λ . With a fixed λ , moreover, viral accuracy \bar{x} can only be attained with positive probability when misleading steady states are also reached with positive probability. By contrast, the following result says if the platform can start with $\lambda = 0$ and switch to $\lambda = 1$ after some time, then there is a limit equilibrium where viral accuracy in later periods is arbitrarily close to \bar{x} with probability arbitrarily close to 1.

Proposition 6. *Suppose $\lambda = 0$ for the first $t_0(n)$ periods and then $\lambda = 1$ for all subsequent periods. We can choose a sequence of $t_0(n)$ such that σ^{maj} is an equilibrium for n sufficiently large, and given strategy σ^{maj} we have $x(n) \rightarrow \bar{x}$ in probability (as the number of agents $n \rightarrow \infty$). The viral accuracy \bar{x} is the highest viral accuracy at any state given any fixed λ and any state-symmetric strategy.*

If the platform were to fix $\lambda = 1$ in all periods, then there is some strictly positive probability that society converges to a misleading steady state. The key idea is that the platform can make the probability of this bad outcome arbitrarily small by showing random news feeds ($\lambda = 0$) to a large

enough number of agents in the early periods. These early agents have a very high probability of generating a viral accuracy higher than $1/2$ because the low virality weight lets independent information accumulate. The platform then exploits this favorable initial condition and switches to $\lambda = 1$ to generate stronger (and very likely correct) beliefs.

The proposition shows that by modifying news-feed algorithms to be dynamic in a simple way, platforms can improve the accuracy of the content they show to users. This policy could provide guidance for regulators concerned about the accuracy of content on platforms. Regulation could, for example, limit how much platforms can show viral content early in the discussion of an issue but then ease these restrictions after some time. The policy is also a potential approach for platforms facing regulatory constraints requiring some level of accuracy.

Real-world platforms of course need not be limited to sampling rules from the particular class we consider in this paper (time-varying or otherwise). The proposition demonstrates that showing less viral content early in a learning process and more viral content later can improve accuracy, and we expect this dynamic would extend to other sampling rules. We note that platforms could obtain very high accuracy by learning the true state and then showing users exclusively stories matching the state. This may be a helpful policy in some situations, but may be controversial or difficult to implement in others. Our result suggests that high accuracy may also be attainable with carefully designed content-neutral algorithms.

4.2 Robustness to Manipulation

We next ask when social learning outcomes on a platform are robust to manipulation. Suppose a manipulator exogenously increases the popularity score of the incorrect stories on the platform in an attempt to mislead. When $\lambda < \lambda^*$, learning outcomes are robust to a small amount of manipulation: a one-time finite increase of some stories' popularity scores does not change steady-state outcomes. Moreover, we can bound the fraction of injected content needed to introduce a misleading steady state. When $\lambda \geq \lambda^*$, such manipulation may be more effective.

To model manipulation, suppose that with probability $1 - \iota$ each agent i behaves as in our baseline model. With probability ι , agent i does not get to play, the story s_i is not posted onto the platform, and we instead increase the total popularity score of the incorrect stories on the platform by $C + 1$ points (if any such stories exist). One interpretation is that ι fraction of the users on

the platform are bots controlled by a manipulator who always shares incorrect stories. We assume the existence of this manipulation is common knowledge, but the following result would also hold if agents are unaware of the manipulation. This model of manipulated content gives an especially simple bound on the robustness of the platform, but one can easily obtain similar bounds under other assumptions.

We define a bound $\underline{\iota} = 1 - \max_{\lambda x + (1-\lambda)q \leq 1/2} \frac{x}{\phi_{\sigma^{\text{maj}}}(x)}$, which depends on λ via the inflow accuracy function $\phi_{\sigma^{\text{maj}}}(x)$. Since $\phi_{\sigma^{\text{maj}}}(x)$ is a polynomial, this bound can be calculated by maximizing a rational function.

Proposition 7. *Suppose $\lambda < \lambda^*$. If $\iota < \underline{\iota}$, there is no misleading steady state at any limit equilibrium.*

The proposition gives a simple lower bound on how much incorrect content can be injected onto a platform without introducing a misleading steady state. When this lower bound is large, we can think of a platform as robust to manipulation campaigns. Conversely, when λ is close to λ^* even small amounts of incorrect content could have large impacts: recall from Corollary 1 that arbitrarily small changes in news feed composition can have substantial impacts on accuracy.

This lower bound is often sharp, though it need not always be. Whether a misleading steady state appears when ι reaches $\underline{\iota}$ depends on whether the majority rule remains a limit equilibrium. On a platform free from manipulation, Theorem 2 tells us the majority rule is the only limit equilibrium when there are only informative steady states. But this may not be true in the presence of manipulated content, as there can be an informative steady state where news feed stories are most likely correct, but they are still less accurate than private signals. So agents will deviate away from the majority rule even before the ι threshold where a misleading steady state appears.

The proof relies on the fact that, as in the proof of Theorem 2, there can only be a misleading steady state for an equilibrium strategy under parameters that also generate a misleading steady state under the majority rule. The remainder of the argument checks for a misleading steady state under the majority rule. Increasing ι is equivalent to rescaling the inflow accuracy function $\phi_{\sigma^{\text{maj}}}(x)$ downward by a constant factor $1 - \iota$, since manipulation never increases the popularity score of the correct stories. This rescaling will create a misleading steady state when there is a solution to $(1 - \iota)\phi_{\sigma^{\text{maj}}}(x) = x$ for x such that $\lambda x + (1 - \lambda)q \leq 1/2$. By straightforward algebra,

the first such solution arises at the value x in this interval maximizing $\frac{x}{\phi_{\sigma^{\text{maj}}}(x)}$, and this happens when $\iota = 1 - x = \underline{\iota}$.

Note that even when ι is not large enough to create a misleading steady state, the manipulation will still decrease viral accuracy at the informative steady state. When the maximum $\max_{\lambda x + (1-\lambda)q \leq 1/2} \frac{x}{\phi_{\sigma^{\text{maj}}}(x)}$ is achieved at the value of x where $\lambda x + (1-\lambda)q = 1/2$, the informative steady state will become misleading at the threshold level of ι . When the maximum is achieved strictly below this x value, however, a new misleading steady state will emerge discontinuously at the threshold level of ι .

When $\lambda \geq \lambda^*$, platforms may be much more vulnerable to manipulation. If users are not aware of the manipulation, then even a finite amount of injected content can make a misleading steady state much more likely if that content arrives sufficiently early. The impact of manipulation is less clear when agents are aware of the manipulations, but outcomes remain sensitive to even small amounts of injected content.

5 Observable Virality

Our baseline model assumes that users do not observe or do not make inferences from the number of times each story in their news feed has been shared. In practice, users on social-media platforms usually do observe information about how many agents have shared, upvoted, or liked posts. In this section, we consider a modification of our model where agents can observe some information about the virality of the stories in their news feeds. In this new environment where agents can discriminate between more and less viral stories, news feeds with enough viral content still generate misleading steady states: we show that every limit equilibrium induces at least one strictly misleading steady state, provided the platform shows a sufficiently large number of viral stories to each agent.

5.1 A Model Where Agents Observe Virality

Allowing agents to fully observe the virality of stories would be an interesting model, but presents apparent technical challenges. The state of the platform would be described by the popularities of all stories, and so the relevant state space grows very large. Similarly, the set of agent observations grows large and strategies become very complicated objects. To avoid these obstacles, we instead

consider a model where agents observe some partial information about stories' virality.

As in the model from Section 2, there is an unknown binary state of nature $\omega \in \{-1, 1\}$. There are n agents randomly placed into a sequence, all starting with the common prior that the two states are equally likely. Each agent i receives a binary private signal about the state, $s_i \in \{-1, 1\}$. Each private signal is a *news story* that matches the state with probability q , where $0.5 < q < 1$.

The platform has two different pools of stories: a *regular story pool* and a *viral story pool*. After the state of nature is drawn but before the first agent acts, the regular story pool is initiated with K_R stories, each matching the state with probability q . The viral story pool is initiated with another set of K_V stories, each matching the state with some probability $q' \in (0, 1)$.

When each agent arrives, they automatically post their news story, adding it to the platform's regular story pool. They also see a *regular news feed* with K_R stories drawn from the regular story pool and a *viral news feed* with K_V stories drawn from the viral story pool. (For simplicity, suppose all sampling is with replacement.) Agents can distinguish between the two different news feeds, so they can tell whether a given news story in their feed comes from the regular story pool or the viral story pool. The agent then shares C stories from the regular news feed and C' stories from the viral story feed, where $C \leq K_R/2$. For each shared story that matches the state, the agent gets utility $u > 0$. Every time a story is shared from the regular news feed, there is some $\alpha > 0$ chance that it *goes viral*: a signal with the same realization is added to the platform's viral story pool. Sharing stories from the viral news feed never creates additional viral stories.

The key difference between this model and the model from Section 2 is that we classify each story on the platform as either "regular" or "viral" and provide each agent with two separate news feeds that contain these two different types of stories. This is a stylized representation of a social-media platform that highlights very popular stories in a special section, in addition to showing users their ordinary news feeds. For example, X displays a selection of "trending" topics in a separate sidebar, distinct from the news feed of regular tweets. Each time agents in our model share a regular news story, there is some (possibly very small) chance that it becomes viral — in the case of X, this corresponds to the algorithm designating a tweet as trending and showing it in the trending sidebar to other users.

5.2 Misleading Steady States with Large Viral Feeds

We begin by adapting the main definitions in our baseline model to the present environment. We can define a mixed strategy in the game to be $\sigma : \{-1, 1\} \times \{0, \dots, K_R\} \times \{0, \dots, K_V\} \rightarrow \Delta(\{0, 1, \dots, C\} \times \{0, 1, \dots, C'\})$, where $\sigma(s, k_R, k_V)$ maps from the realization of the private signal, the number of positive stories in the regular news feed, and the number of positive stories in the viral news feed to the possibly random numbers of positive stories shared from the regular feed and the viral feed. Strategies must satisfy the relevant feasibility constraints in terms of the numbers of available positive and negative stories in the two news feeds.

We focus on symmetric Bayesian Nash equilibria where all players use the same strategy and this common strategy treats positive and negative stories symmetrically, which we call symmetric equilibria. For fixed parameters $q, q', K_R, K_V, C, C', \alpha$, a mixed strategy σ^* is a *limit equilibrium* if there exists a sequence of symmetric equilibria $(\sigma^{(j)})_{j=1}^\infty$ for finite societies with these parameters and n_j agents, where $n_j \rightarrow \infty$ and $\lim_{j \rightarrow \infty} \sigma^{(j)} = \sigma^*$.

We define *viral accuracy* after person t moves as the fraction of stories in the viral story pool that match the state, denoted by $x(t)$. Suppose all individuals in an infinite society use the same state-symmetric strategy σ . Then we get the stochastic process of viral accuracy $(x(t))_{t=1}^\infty$, and we show variants of Proposition 2 and Theorem 1 in the appendix. We call a point x^* such that $x(t) \rightarrow x^*$ with positive probability a *steady state* induced by σ , and this steady state is called *misleading* if $x^* < 1/2$.

The main message of Theorem 2 is that when news feeds show enough viral content, misleading steady states will arise. The next result shows this continues to hold when agents partially observe which stories are viral: as K_V the size of the viral news feed grows sufficiently large, every limit equilibrium must induce a misleading steady state where a strict majority of the stories in the platform's viral story pool are wrong. We now vary the composition of news feeds by changing the size of the viral news feed rather than the virality weight parameter λ in the model from Section 2. A larger K_V and a larger λ both lead to misleading steady states. So the overall lesson is that rational agents can generate misleading steady states on platforms where the news-feed contents depend strongly on others' sharing, even when correct information arrives on the platform in every period and even if agents make some distinctions between viral and non-viral stories.

Theorem 3. *Fix q, q', C, C', α . For any K_R large enough so that $\mathbb{P}[\text{Binom}(K_R, 1 - q) \geq C] > 1/2$, there exists some \bar{K}_V so that whenever $K_V \geq \bar{K}_V$, every limit equilibrium induces at least one misleading steady state.*

An additional assumption needed for Theorem 3 is that the regular news feed is large enough to have at least a 50% chance of showing each agent C or more regular news stories that mismatch the state. Similar to our model from Section 2, in this model misleading steady states arise from rational agents sometimes sharing wrong regular stories, so we need to ensure enough of these wrong stories exist in the regular news feed to be potentially shared and go viral.

6 Conclusion

We have developed a model of learning from social media where rational users share stories that they believe are more likely to match the state. Users see news feeds, which depend on what stories others share as well as how much weight the platform’s sampling algorithm places on these sharing decisions. Showing more viral stories can help aggregate more information. But at a critical threshold, a misleading steady state where users primarily see and share incorrect stories discontinuously emerges.

We have focused on social-media platforms, but conclude by mentioning two other applications of our analysis. First, our model could also be interpreted as describing offline information-sharing dynamics. The parameter λ would then measure how frequently people communicate their personal experiences or private information relative to passing along others’ experiences or information. Second, related models have been used to describe product adoption dynamics when consumers use simple heuristics (Smallwood and Conlisk, 1979). Our techniques suggest a path toward introducing equilibrium behavior into such models.

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A Proofs

A.1 Proof of Proposition 1

Proof. Fixing n gives a symmetric finite game. So by Kakutani's fixed point theorem, there exists a symmetric BNE.

Now fix q, K, C , and λ . For each n , there exists a symmetric BNE $\sigma^{(n)}$. Because the space of strategies σ is compact, we can choose a convergent subsequence. The limit of this subsequence is a limit equilibrium. \square

A.2 Proof of Proposition 2

Proof. The proof applies a convergence result from stochastic approximation from Chapter 2 of [Borkar \(2023\)](#). Suppose agents use strategy σ . Without loss of generality, we can condition on $\omega = 1$.

Let

$$Y = \{\mathbf{y} = (x, z) \in [0, 1]^2\}.$$

Recall $\rho_t(s_i)$ is the popularity of signal s_i after agent t acts. For each t , define $\mathbf{y}(t) \in Y$ by

$$x(t) = \frac{\sum_{s_i=1} \rho_t(s_i)}{(C+1)t} \text{ and } z(t) = \frac{\sum_{s_i=1} 1}{t}.$$

The first entry of $\mathbf{y}(t) = (x(t), z(t))$ measures the fraction of shares which are shares of signals with realization 1 up to time t . The second entry measures the fraction of private signals which have realization 1 up to time t .

We can write

$$\mathbf{y}(t+1) = \mathbf{y}(t) + \frac{1}{t+1} (\xi(t+1) - \mathbf{y}(t)),$$

where $\xi(t+1)$ is the random variable with first entry equal to the fraction of shared signals which have realization 1 in period $t+1$ and second entry is a binary indicator for whether $s_{t+1} = 1$.

Following the notation of [Borkar \(2023\)](#), we write

$$h(\mathbf{y}(t)) = \mathbb{E}[\xi(t+1) | \mathbf{y}(t)] - \mathbf{y}(t) \text{ and } M(t+1) = \xi(t+1) - \mathbb{E}[\xi(t+1) | \mathbf{y}(t)].$$

We can then decompose the change in the stochastic process $\mathbf{y}(t)$ into $h(\mathbf{y}(t))$, which depends deterministically on $\mathbf{y}(t)$, and $M(t+1)$, which is a martingale:

$$\mathbf{y}(t+1) = \mathbf{y}(t) + \frac{1}{t+1} (h(\mathbf{y}(t)) + M(t+1)).$$

We would like to apply Theorem 2.1 of Chapter 2 of [Borkar \(2023\)](#), which requires the following assumptions:

(A1) h is Lipschitz continuous.

(A2) $\sum_t \frac{1}{t+1} = \infty$ while $\sum_t \frac{1}{(t+1)^2} < \infty$.

(A3) $\mathbb{E}[M(t+1) \mid \mathbf{y}(t)] = 0$ and $\{M(t)\}$ are square-integrable with

$$\mathbb{E}[\|M(t+1)\|^2 \mid \mathbf{y}(t)] \leq \kappa(1 + \|\mathbf{y}(t)\|^2)$$

a.s. for all n and some $\kappa > 0$.

(A4) $\|\mathbf{y}(t)\|$ remains bounded a.s.

Properties **(A2)** and **(A4)** are immediate. For **(A3)**, the martingale property holds by the construction of $M(t)$ and the remaining properties hold because $M(t)$ is bounded (independent of t).

Property **(A1)** remains. Since $-\mathbf{y}(t)$ is Lipschitz continuous in $\mathbf{y}(t)$, we must check that $\mathbb{E}[\xi(t+1) \mid \mathbf{y}(t)]$ is Lipschitz continuous in $\mathbf{y}(t)$.

Write $\sigma_1(s, k)$ for the expected number of “1” signals that strategy σ shares, when the agent’s private signal is s and k signals in the sample are “1”. Write $P_k(x, z, \lambda)$ for $\mathbb{P}[\text{Binom}(K, \lambda x + (1 - \lambda)z) = k]$, where $\text{Binom}(k, p)$ is the binomial distribution with k trials and success probability p .

Then for $t \geq K$, the conditional expectation of the random variable $\xi_1(t+1)$ is equal to

$$\frac{1}{C+1} \left(q + \sum_{0 \leq k \leq K} P_k(x(t), z(t), \lambda) (q\sigma_1(1, k) + (1-q)\sigma_1(-1, k)) \right).$$

This is a polynomial of degree at most K in $x(t)$ and $z(t)$, and therefore is Lipschitz continuous on Y . The conditional expectation of $\xi_2(t+1)$ is constant, and therefore Lipschitz continuous in Y as

well.

For $\mathbf{r} = (r_1, r_2)$, we can define a continuous-time differential equation by letting

$$\dot{\mathbf{r}}(t) = h(\mathbf{r}(t)), t \geq 0. \quad (1)$$

An invariant set A of (1) is a set such that $\mathbf{r}(0) \in A$ implies $\mathbf{r}(t) \in A$ for all $t \geq 0$. An invariant set is internally chain transitive if for any $\mathbf{r}, \mathbf{r}' \in A$, $\epsilon > 0$ and $T > 0$, there exists $\mathbf{r}^0 = \mathbf{r}, \mathbf{r}^1, \dots, \mathbf{r}^n = \mathbf{r}' \in A$ such that the trajectory of $\mathbf{r}(t)$ starting from $\mathbf{r}(0) = \mathbf{r}^i$ meets with an ϵ -neighborhood of \mathbf{r}^{i+1} at some time $t \geq T$.

By Theorem 2.1 of Chapter 2 of [Borkar \(2023\)](#), the stochastic process $\mathbf{y}(t)$ converges to an internally chain transitive invariant set of equation (1). Because $r_2(t) \rightarrow q$, any internally chain transitive invariant set must be contained in $[0, 1] \times \{q\}$. We claim that at any \mathbf{r} contained in an internally chain transitive invariant set A , we must have

$$\frac{dr_1(t)}{dt} = 0$$

when $\mathbf{r}(t) = \mathbf{r}$. Suppose an internally chain transitive invariant set A of (1) contains a point \mathbf{r} at which

$$\frac{dr_1(t)}{dt} > 0.$$

Letting $\mathbf{r}(0) = \mathbf{r}$, we can choose some $t' > 0$ such that $r_1(t') > r_1(0)$ and

$$\frac{dr_1(t)}{dt} > 0$$

at $t = t'$. Now let $\mathbf{r}' = \mathbf{r}(t')$. We have $\mathbf{r}' \in A$ by invariance.

If we consider the trajectory $\mathbf{r}(t)$ beginning with $\mathbf{r}(0) = \mathbf{r}'$, we cannot have $r_1(t)$ fall below $r_1(0)$ since $\dot{r}_1(0) > 0$ and the sign of $\dot{r}_1(t)$ only depends on t through $r_1(t)$. For $\epsilon > 0$ sufficiently small this implies that the trajectory $\mathbf{r}(t)$ beginning with $\mathbf{r}(0) = \mathbf{r}'$ never enters an ϵ -neighborhood of \mathbf{r} . This contradicts the assumption that A is internally chain transitive.

If A contains a point \mathbf{r} at which

$$\frac{dr_1(t)}{dt} < 0,$$

we obtain a contradiction similarly. This shows the claim that

$$\frac{dr_1(t)}{dt} = 0$$

at all \mathbf{r} contained in an internally chain transitive invariant set.

Values of $r_1(t)$ for which $\frac{dr_1(t)}{dt} = 0$ correspond to the roots of a non-linear polynomial, and therefore there are at most finitely many such values. Calling the set of such values $X^*(1)$, we can conclude that $x(t)$ converges almost surely to $x^* \in X^*(1)$. \square

A.3 Proof of Theorem 1

We say a fixed point x^* of $\phi_\sigma(x)$ is a *touchpoint* if there exists $\epsilon > 0$ such that $\phi_\sigma(x) < x$ for all $x \neq x^*$ in $(x^* - \epsilon, x^* + \epsilon)$ or $\phi_\sigma(x) > x$ for all $x \neq x^*$ in $(x^* - \epsilon, x^* + \epsilon)$.

Case (i): x^* is a touchpoint.

The proof extends the arguments from Theorem 1 of Pemantle (1991). Suppose that $\phi_\sigma(x) > x$ for all $x \neq x^*$ in $(x^* - \epsilon, x^* + \epsilon)$. The other case is the same.

Fix $v \in (0, \frac{1}{2})$ and $v_1 \in (v, \frac{1}{2})$. Choose $\gamma > 1$ such that $\gamma v_1 < \frac{1}{2}$. Define $g(r) = re^{(1-r)/(2v_1\gamma)}$. Then $g(1) = 1$ and $g'(1) = 1 - 1/(2v_1\gamma) < 0$, so we can choose $r_0 \in (0, 1)$ with $g(r_0) > 1$. Also define

$$T(n) = e^{n(1-r_0)/(\gamma v_1)}.$$

Then

$$g(r_0)^n = r_0^n T(n)^{1/2} > 1.$$

Choose N such that $\gamma r_0^N < \epsilon$. Since $T(1)^{1/2} r_0 = g(r_0) > 1$, we can find $\alpha > 0$ such that $T(1)^{1/2-\alpha} > r_0$ and therefore $T(n)^{1/2-\alpha} r_0^{-n} \rightarrow \infty$. Let $\tau_N = \inf\{j > T(N) : x(j-1) < x^* - r_0^N < x(j)\}$ (using the convention that $\tau_N = -\infty$ if there is no such j). For each $n \geq N$, define

$$\tau_{n+1} = \inf\{j \geq \tau_n : x(j) > x^* - r_0^{n+1}\}.$$

So τ_n is the first time the stochastic process crosses $x^* - r_0^n$.

We will show the probability that $\tau_n > T(n)$ for all $n \geq N$ is positive. Since $x(t) \rightarrow x^*$ from

below whenever $\tau_n > T(n)$ for all $n \geq N$, this will complete the case.

Let $z(t)$ be the fraction of private signals up to period t with realization $s_i = 1$. We first bound the probability that $z(t)$ is far from q . Define a function

$$\phi_{\sigma,z}(x) := \frac{q + \sum_{k=0}^K \mathbb{P}[\text{Binom}(K, \lambda x + (1-\lambda)z) = k] \cdot [q \cdot \sigma(1, k) + (1-q) \cdot \sigma(-1, k)]}{1 + C}$$

to be the inflow accuracy when a fraction z of past private signals have value 1.

We begin by defining an event \mathcal{C} under which the number of private signals with positive realization is close to q for t sufficiently large. Let \mathcal{C}_1 be the event that for all $n \geq N$ and for all $t \geq T(n)$,

$$\phi_{\sigma,z(t)}(x) - x \geq -1/T(n)^{1/2-\alpha}$$

on $(x^* - \epsilon, x^* + \epsilon)$. Because $\phi_{\sigma,z}(x) - x$ is polynomial (in z and x) and is non-negative on this interval when $z = q$, this holds for $|z(t) - q| < B/T(n)^{1/2-\alpha}$ for some $B > 0$.

Suppose event \mathcal{C}_1 holds and $\tau_n > T(n)$. Then we have

$$\begin{aligned} \sum_{t=\tau_n}^j h_1(\mathbf{y}(t))/(t+1) &= \sum_{t=\tau_n}^j (\phi_{\sigma,z(t)}(x(t)) - x(t))/(t+1) \\ &\geq - \sum_{m=n}^{\infty} \frac{1}{T(m)^{1/2-\alpha}} \sum_{T(m) \leq t < T(m+1)} \frac{1}{t+1} \text{ by the definition of } \mathcal{C}_1 \\ &\geq - \sum_{m=n}^{\infty} \frac{\log(\lceil T(m+1) \rceil) - \log(\lceil T(m) \rceil)}{T(m)^{1/2-\alpha}} \\ &\geq - \sum_{m=n}^{\infty} \left(\frac{1-r_0}{\gamma v_1} + 1 \right) \cdot e^{-m(1/2-\alpha)(1-r_0)/(\gamma v_1)} \\ &= - \left(\frac{1-r_0}{\gamma v_1} + 1 \right) \cdot \frac{e^{-n(1/2-\alpha)(1-r_0)/(\gamma v_1)}}{1 - e^{-(1/2-\alpha)(1-r_0)/(\gamma v_1)}}. \end{aligned} \tag{2}$$

We define $\mu = \left(\frac{1-r_0}{\gamma v_1} + 1 \right) \cdot \frac{1}{1 - e^{-(1/2-\alpha)(1-r_0)/(\gamma v_1)}}$, so that the right-hand side is $-\mu T(n)^{-(1/2-\alpha)}$.

Let \mathcal{C}_2 be the event that for all $n \geq N$ and for all $t \geq T(n)$,

$$\phi_{\sigma,z(t)}(x) - x \leq v\gamma r_0^n \tag{3}$$

for all $x \in [x^* - \gamma r_0^n, x^*]$. Because $\phi_{\sigma,z}(x) - x$ is polynomial (in z and x) and

$$\frac{d(\phi_{\sigma,q}(x) - x)}{dx}(x^*) = 0,$$

we can choose B' such that for all $n \geq N$ we have

$$\phi_{\sigma,z}(x) - x \leq v\gamma r_0^n$$

for $x \in [x^* - \gamma r_0^n, x^*]$ whenever $|z - q| < B'r_0^n$ (since we can bound the entries of the Hessian of $\phi_{\sigma,z}(x) - x$ above by a constant on the rectangle $[x^* - \gamma r_0^N, x^*] \times [q - r_0^N, q + r_0^N]$). Because $r_0^n > T(n)^{1/2-\alpha}$, this holds for $|z(t) - q| < B'/T(n)^{1/2-\alpha}$ for some $B' > 0$.

Define the event $\mathcal{C} = \mathcal{C}_1 \cap \mathcal{C}_2$ to be the intersection of these two events. The event \mathcal{C} holds when $|z(t) - q| < \min(B, B')/T(n)^{1/2-\alpha}$ for all $n \geq N$ and all $t \geq T(n)$. By the Chernoff bound and the inequalities $t \geq T(n)$ and $q > 1 - q$, the probability of $|z(t) - q| > \min(B, B')/T(n)^{1/2-\alpha}$ is at most $2e^{-\min(B, B')^2 t^{2\alpha}/(2q^2)}$. So the probability that the event \mathcal{C} does not hold for some $n \geq N$ and all $t \geq T(n)$ is at most

$$2 \sum_{n=N}^{\infty} \sum_{t=T(n)}^{\infty} 2e^{-\min(B, B')^2 t^{2\alpha}/(2q^2)}.$$

For N sufficiently large, this sum is approximately

$$\sum_{n=N}^{\infty} \frac{1}{\alpha} \left(\frac{\min(B, B')^2}{2q^2} \right)^{-\frac{1}{2\alpha}} \Gamma \left(\frac{1}{2\alpha}, T(n)^{2\alpha} \min(B, B')^2/(2q^2) \right)$$

where $\Gamma(s, x)$ is the incomplete Gamma function. Since $\Gamma(s, x)/(x^{s-1}e^{-x}) \rightarrow 1$ as $x \rightarrow \infty$, this sum converges to zero as $N \rightarrow \infty$. Increasing N if necessary, we can conclude that the event \mathcal{C} has positive probability. For the remainder of case (i), we condition on this event \mathcal{C} .

Now let \mathcal{B} be the event $\{\inf_{j > \tau_n} x(j) \geq x^* - \gamma r_0^n\}$. We will bound the probability of this event conditional on $\tau_n > T(n)$. Let $Z_{m,n} = \sum_{t=m}^{n-1} M(t+1)$ be the sum of the martingale parts of the stochastic process. Because the differences $M(t)$ are martingales with $|M(t)| \leq (C+1)/(t+1)$, we have

$$\mathbb{E}[Z_{m,\infty}^2] = \sum_{t=m}^{\infty} \mathbb{E}[M(t)^2] \leq \sum_{t=m}^{\infty} \left(\frac{C+1}{t+1} \right)^2 \leq \frac{(C+1)^2}{m}. \quad (4)$$

We have:

$$\begin{aligned}
\mathbb{P}[\mathcal{B}^c \mid \tau_n > T(n)] &= \mathbb{P}\left[\inf_{j > \tau_n} x(j) < x^* - \gamma r_0^n \mid \tau_n > T(n)\right] \\
&\leq \mathbb{P}\left[\inf_{j > \tau_n} Z_{\tau_n, j} < -(\gamma - 1)r_0^n + \mu T(n)^{-(1/2-\alpha)} \mid \tau_n > T(n)\right] \text{ by equation (2)} \\
&\leq \mathbb{E}\left[Z_{\tau_n, \infty}^2 \mid \tau_n > T(n)\right] / ((\gamma - 1)r_0^n - \mu T(n)^{-(1/2-\alpha)})^2 \text{ by Chebyshev's inequality} \\
&\leq (C + 1)^2 e^{-n(1-r_0)/(v_1\gamma)} ((\gamma - 1)r_0^n - \mu T(n)^{-(1/2-\alpha)})^{-2} \\
&\text{by inequality (4) and the definition of } T(n).
\end{aligned}$$

Recall that $T(n)^{1/2-\alpha} r_0^{-n} \rightarrow \infty$, so for n sufficiently large

$$(\gamma - 1)r_0^n - \mu T(n)^{-(1/2-\alpha)} \geq \frac{\gamma - 1}{2} r_0^n.$$

We conclude that

$$\mathbb{P}[\mathcal{B}^c \mid \tau_n > T(n)] \leq (C + 1)^2 \left(\frac{\gamma - 1}{2}\right)^{-2} g(r_0)^{-2n}.$$

This bounds the conditional probability of the event \mathcal{B} not holding.

When the event \mathcal{B} does hold and $\tau_n > T(n)$,

$$\begin{aligned}
\sum_{\substack{T(n) < t < T(n+1) \\ x(t) < x^*}} h_1(\mathbf{y}(t)) / (t + 1) &= \sum_{\substack{T(n) < t < T(n+1) \\ x(t) < x^*}} (\phi_{\sigma, z(t)}(x(t)) - x(t)) / (t + 1) \\
&\leq \sum_{\substack{T(n) < t < T(n+1) \\ x(t) < x^*}} v\gamma r_0^n / (t + 1) \text{ by equation (3)} \\
&\leq (\log \lceil T(n + 1) \rceil - \log \lceil T(n) \rceil) (v\gamma r_0^n) \\
&\text{by the partial sums of the harmonic series} \\
&\leq (v\gamma r_0^n) ((1 - r_0) / (\gamma v_1) + 1 / T(n)) \\
&= (v / v_1) (r_0^n - r_0^{n+1}) + v\gamma r_0^n / T(n).
\end{aligned}$$

Now suppose \mathcal{B} holds and $\tau_n > T(n)$ but $\tau_{n+1} \leq T(n+1)$. Then

$$\begin{aligned}
Z_{\tau_n, \tau_{n+1}} &= x(\tau_{n+1}) - x(\tau_n) - \sum_{t=\tau_n}^{\tau_{n+1}-1} h_1(\mathbf{y}(t))/(t+1) \\
&\geq x(\tau_{n+1}) - x(\tau_n) - \sum_{\substack{T(n) < t < T(n+1) \\ x(t) < x^*}} h_1(\mathbf{y}(t))/(t+1) \\
&\geq r_0^n - r_0^{n+1} - \xi_n - (v/v_1)(r_0^n - r_0^{n+1}) - v\gamma r_0^n/T(n) \text{ by the inequality above and definition of } \tau_n \\
&= r_0^n(1-r_0)(1-v/v_1) - \xi_n - v\gamma r_0^n/T(n),
\end{aligned}$$

where ξ_n is an error term since $x(\tau_n)$ may be larger than $x^* - r_0^n$ and $\tilde{\xi}_n = \xi_n + v\gamma r_0^n/T(n)$. Since the error term ξ_n is at most $1/T(n)$ and therefore is lower order than r_0^n , we have

$$\frac{r_0^n(1-r_0)(1-v/v_1) - \tilde{\xi}_n}{r_0^n(1-r_0)(1-v/v_1)} \rightarrow 1 \quad (5)$$

as $n \rightarrow \infty$.

Combining our bounds, we have:

$$\begin{aligned}
\mathbb{P}[\tau_{n+1} \leq T(n+1) \mid \tau_n > T(n)] &\leq \mathbb{P}[\mathcal{B}^c \mid \tau_n > T(n)] + \\
&\quad \mathbb{P}\left[\mathcal{B}, \sup_{\tau_{n+1}} Z_{\tau_n, \tau_{n+1}} \geq r_0^n(1-r_0)(1-v/v_1) - \tilde{\xi}_n \mid \tau_n > T(n)\right] \\
&\leq (C+1)^2 \left(\frac{\gamma-1}{2}\right)^{-2} g(r_0)^{-2n} + \frac{\mathbb{E}[Z_{\tau_n, \infty}^2 \mid \tau_n > T(n)]}{(r_0^n(1-r_0)(1-v/v_1) - \tilde{\xi}_n)^2} \\
&\quad \text{by Chebyshev's inequality} \\
&\leq (C+1)^2 \left(\frac{\gamma-1}{2}\right)^{-2} g(r_0)^{-2n} + \frac{(C+1)^2 T(n)^{-1}}{(r_0^n(1-r_0)(1-v/v_1) - \tilde{\xi}_n)^2} \\
&\quad \text{by inequality (4)} \\
&\leq (C+1)^2 \left(\frac{\gamma-1}{2}\right)^{-2} g(r_0)^{-2n} + \\
&\quad (C+1)^2 ((1-r_0)(1-v/v_1))^{-2} g(r_0)^{-2n} \cdot \frac{r_0^n(1-r_0)(1-v/v_1) - \tilde{\xi}_n}{r_0^n(1-r_0)(1-v/v_1)}.
\end{aligned}$$

We claim that the sum of these probabilities converges. The sum of the first terms converges because $g(r_0) > 1$. For the second term, recall that the fraction $\frac{r_0^n(1-r_0)(1-v/v_1) - \tilde{\xi}_n}{r_0^n(1-r_0)(1-v/v_1)}$ converges to 1.

So the sum of the second terms also converges because $g(r_0) > 1$.

We have

$$\mathbb{P}[\tau_n > T(n) \text{ for all } n \geq N] = \mathbb{P}[\tau_N > T(N)] \prod_{n=N}^{\infty} (1 - \mathbb{P}[\tau_{n+1} \leq T(n+1) \mid \tau_n > T(n)]).$$

On the right-hand side, each factor in the product is positive and

$$\sum_{n=N}^{\infty} \mathbb{P}[\tau_{n+1} \leq T(n+1) \mid \tau_n > T(n)]$$

is finite. By a standard result on infinite products, this implies the product is positive. So the probability that $\tau_n > T(n)$ for all $n \geq N$ is positive, which implies that the probability $\pi(x^*|\sigma)$ of converging to x^* is positive.

Case (ii): There exists $\epsilon > 0$ such that $\phi_\sigma(x) > x$ for all $x \in (x^* - \epsilon, x^*)$ and $\phi_\sigma(x) < x$ for all $x \in (x^*, x^* + \epsilon)$.

Our argument is based on the related result for generalized Pólya urns from [Hill, Lane, and Sudderth \(1980\)](#). We begin with a lemma, which says that suitably changing a stochastic process away from a neighborhood of a fixed point does not affect whether we converge to that fixed point with positive probability:

Lemma 2. *Suppose*

$$\tilde{\mathbf{y}}(t+1) = \tilde{\mathbf{y}}(t) + \frac{1}{t+1} \left(\tilde{\xi}(t+1) - \tilde{\mathbf{y}}(t) \right),$$

where the conditionally i.i.d. random variables $\tilde{\xi}(t+1)$ have the same conditional distribution as $\xi(t+1)$ in a neighborhood U of (x^, q) , have the same support as $\xi(t+1)$ for all $(x, z) \in (0, 1)^2$, and have expectations $\mathbb{E}[\tilde{\xi}(t+1)]$ that are Lipschitz continuous in (x, z) . Then $x(t)$ converges to x^* with positive probability if and only if $\tilde{x}(t) = \tilde{\mathbf{y}}_1(t)$ does.*

Proof. The stochastic process $x(t)$ converges to x^* with positive probability if and only if there exists some T and some $(x(T), z(T))$ reached with positive probability under $\mathbf{y}(t)$ such that starting with initial condition $(x(T), z(T))$, with positive probability $x(t) \rightarrow x^*$ and $(x(t), z(t)) \in U$ for $t \geq T$.

Because the random variables $\tilde{\xi}(t)$ have the same support as $\xi(t)$ whenever x and z are interior, the point $(x(T), z(T))$ is reached with positive probability under $\tilde{\mathbf{y}}(t)$ if and only if it is reached with

positive probability under $\mathbf{y}(t)$. Because $\tilde{\xi}(t)$ and $\xi(t)$ agree on U , starting with initial condition $(x(T), z(T))$, with positive probability $\tilde{x}(t) \rightarrow x^*$ and $(\tilde{x}(t), \tilde{z}(t)) \in U$ for $t \geq T$ if and only if the same holds for $(x(t), z(t))$. These conditions hold for some $(x(T), z(T))$ if and only if $\tilde{x}(t)$ converges to x^* with positive probability. \square

Now choose $\tilde{\xi}(t)$ satisfying the conditions of the lemma, agreeing with $\xi(t)$ in the second coordinate, and such that the unique fixed point of the corresponding function $\tilde{\phi}_\sigma(x)$ is x^* . To do so, choose an open neighborhood U of (x^*, q) such that x^* is the unique fixed point of $\phi_\sigma(x)$ with $(x, q) \in \bar{U}$. Let $\tilde{\xi}(t) = \xi(t)$ on the closure \bar{U} of U . For each z , let $\tilde{\xi}(t)$ be constant in x outside of the neighborhood U .

Then $\tilde{\xi}(t)$ and $\xi(t)$ have the same support for all interior x and z . Lipschitz continuity follows from Lipschitz continuity of the expectations of $\xi(t)$ in x and z , which we checked in the proof of Proposition 2.

Since x^* is the unique fixed point of $\phi_\sigma(x)$, by the same argument as in Proposition 2, we have $\tilde{x}(t) \rightarrow x^*$ almost surely. Note that this step uses Lipschitz continuity of $\mathbb{E}[\tilde{\xi}(t+1)]$. So by Lemma 2, $x(t) \rightarrow x^*$ with positive probability.

A.4 Proof of Lemma 1

Proof. If $x > 1/2$, then sampling accuracy is $\lambda x + (1-\lambda)q > 1/2$ since $q > 1/2$ also. If $x < 1/2$ and $\phi_{\sigma^{\text{maj}}}(x) = x$, then the sampling accuracy must be strictly less than $1/2$. Otherwise, if sampling accuracy is $1/2$ or higher, then the majority rule implies $\sum_{k=0}^K P_k(x, \lambda) \cdot [q \cdot \sigma^{\text{maj}}(1, k) + (1-q) \cdot \sigma^{\text{maj}}(-1, k)] \geq C/2$ and so $\phi_{\sigma^{\text{maj}}}(x) > 1/2$. Finally, if $x = 1/2$ were a steady state, then its sampling accuracy would be at least $1/2$, so again by the same reason we would have $\phi_{\sigma^{\text{maj}}}(x) > 1/2$, a contradiction. \square

A.5 Proof of Theorem 2

A.5.1 Preliminary Lemmas

We first state and prove three preliminary lemmas.

Lemma 3. *Suppose σ is state symmetric, $\mathbb{E}[\sigma(1, k)] \geq \mathbb{E}[\sigma(-1, k)]$ for every $0 \leq k \leq K$, and that $\sigma(1, K/2)(C) = 1$, $\sigma(-1, K/2)(0) = 1$ if K is even. If sampling accuracy at x is weakly smaller than*

$1/2$, then $\phi_{\sigma^{\text{maj}}}(x) \leq \phi_{\sigma}(x)$. If sampling accuracy at x is strictly smaller than $1/2$ and $\sigma \neq \sigma^{\text{maj}}$, then $\phi_{\sigma^{\text{maj}}}(x) < \phi_{\sigma}(x)$.

Proof. We have

$$\phi_{\sigma^{\text{maj}}}(x) = \frac{q + P_{K/2}(x, \lambda) \cdot q \cdot C + \sum_{k > K/2} P_k(x, \lambda) \cdot C}{1 + C}$$

We first note that in a news feed with $K/2$ out of K positive stories, by assumption on σ it must share C positive stories that match the private signal, so both strategies contribute $q \cdot C$ correct shares in expectation.

For each $k > K/2$, by symmetry we have $\mathbb{E}[\sigma(1, k)] = C - \mathbb{E}[\sigma(-1, K - k)]$ and $\mathbb{E}[\sigma(-1, k)] = C - \mathbb{E}[\sigma(1, K - k)]$.

We have

$$\begin{aligned} & P_k(x, \lambda) \cdot (q \cdot \mathbb{E}[\sigma(1, k)] + (1 - q) \cdot \mathbb{E}[\sigma(-1, k)]) + P_{K-k}(x, \lambda) \cdot (q \cdot \mathbb{E}[\sigma(1, K - k)] + (1 - q) \cdot \mathbb{E}[\sigma(-1, K - k)]) \\ &= P_k(x, \lambda) \cdot (q \cdot (C - \mathbb{E}[\sigma(-1, K - k)]) + (1 - q) \cdot (C - \mathbb{E}[\sigma(1, K - k)])) \\ & \quad + P_{K-k}(x, \lambda) \cdot (q \cdot \mathbb{E}[\sigma(1, K - k)] + (1 - q) \cdot \mathbb{E}[\sigma(-1, K - k)]) \\ &= P_k(x, \lambda) \cdot C - P_k(x, \lambda) \cdot (q \mathbb{E}[\sigma(-1, K - k)] + (1 - q) \mathbb{E}[\sigma(1, K - k)]) \\ & \quad + P_{K-k}(x, \lambda) \cdot (q \cdot \mathbb{E}[\sigma(1, K - k)] + (1 - q) \cdot \mathbb{E}[\sigma(-1, K - k)]) \\ &\geq P_k(x, \lambda) \cdot C - P_k(x, \lambda) \cdot (q \mathbb{E}[\sigma(-1, K - k)] + (1 - q) \mathbb{E}[\sigma(1, K - k)]) \\ & \quad + P_{K-k}(x, \lambda) \cdot ((1 - q) \cdot \mathbb{E}[\sigma(1, K - k)] + q \cdot \mathbb{E}[\sigma(-1, K - k)]) \end{aligned}$$

using the fact that $q > 1/2$ and $\mathbb{E}[\sigma(1, K - k)] \geq \mathbb{E}[\sigma(-1, K - k)]$ by the hypothesis on σ .

Suppose x is weakly misleading, so $\lambda x + (1 - \lambda)q \leq 1/2$. Then $P_{K-k}(x, \lambda) \geq P_k(x, \lambda)$ since $k > K/2$. This shows when $\omega = 1$, the expected number of positive stories shared by σ with a k majority in the news feed is weakly larger than that shared by σ^{maj} . So $\phi_{\sigma^{\text{maj}}}(x) \leq \phi_{\sigma}(x)$.

Now suppose x strictly misleading, so $\lambda x + (1 - \lambda)q < 1/2$. Then $P_k(x, \lambda) < P_{K-k}(x, \lambda)$ and the final term is strictly larger than $P_k(x, \lambda) \cdot C$ except when $\mathbb{E}[\sigma(-1, K - k)] = \mathbb{E}[\sigma(1, K - k)] = 0$. This shows we get $\phi_{\sigma^{\text{maj}}}(x) < \phi_{\sigma}(x)$ except when $\sigma(-1, k) = \sigma(1, k)$ is the degenerate distribution on 0 for any $k < K/2$, which by symmetry only happens when $\sigma = \sigma^{\text{maj}}$. \square

Lemma 4. Suppose σ is state symmetric and $\sigma(1, k)(C) = 1$ for every $k \geq K/2$. Then, ϕ_{σ} does

not have any fixed point x with $\lambda x + (1 - \lambda)q > 1/2$ and $x \leq q$.

Proof. Suppose by way of contradiction that such a fixed point x exists. Let $y = \lambda x + (1 - \lambda)q$ be the sampling accuracy, and note $x \leq y \leq q$, with $y > 1/2$. Expected number of positive stories shared by each new agent when $\omega = 1$ is:

$$q \cdot \sum_{k=0}^K P_k(x, \lambda) \cdot \mathbb{E}[\sigma(1, k)] + (1 - q) \cdot \sum_{k=0}^K P_k(x, \lambda) \cdot \mathbb{E}[\sigma(-1, k)].$$

We know $\mathbb{E}[\sigma(1, k)] = C \geq \mathbb{E}[\sigma(-1, k)]$ for each $k \geq K/2$, and conversely $\mathbb{E}[\sigma(-1, k)] = 0 \leq \mathbb{E}[\sigma(1, k)]$ for each $k < K/2$. This means $\sum_{k=0}^K P_k(x, \lambda) \cdot \mathbb{E}[\sigma(1, k)] \geq \sum_{k=0}^K P_k(x, \lambda) \cdot \mathbb{E}[\sigma(-1, k)]$. Since $q \geq y > 1/2$, this expected number of shared positive stories is at least

$$y \cdot \sum_{k=0}^K P_k(x, \lambda) \cdot \mathbb{E}[\sigma(1, k)] + (1 - y) \cdot \sum_{k=0}^K P_k(x, \lambda) \cdot \mathbb{E}[\sigma(-1, k)].$$

For each $0 \leq k < K/2$,

$$\begin{aligned} (1 - y) \cdot P_{K-k}(x, \lambda) &= (1 - y) \cdot \binom{K}{k} y^{K-k} (1 - y)^k \\ &\geq y \cdot \binom{K}{k} y^k (1 - y)^{K-k}, \end{aligned}$$

so

$$(1 - y) \cdot P_{K-k}(x, \lambda) \cdot \mathbb{E}[\sigma(-1, K - k)] \geq y \cdot P_k(x, \lambda) \cdot \mathbb{E}[\sigma(-1, K - k)]$$

and the inequality is strict for $k = 0$ because $y > 1/2$ and $\sigma(-1, K)$ must share C positive stories as there are no negative stories in the news feed. So we have

$$\begin{aligned} &y \cdot \sum_{k=0}^K P_k(x, \lambda) \cdot \mathbb{E}[\sigma(1, k)] + (1 - y) \cdot \sum_{k=0}^K P_k(x, \lambda) \cdot \mathbb{E}[\sigma(-1, k)] \\ &= y \cdot \sum_{k \geq K/2} P_k(x, \lambda) \cdot C + y \cdot \sum_{k < K/2} P_k(x, \lambda) \cdot \mathbb{E}[\sigma(1, k)] + (1 - y) \cdot \sum_{k > K/2} P_k(x, \lambda) \cdot \mathbb{E}[\sigma(-1, k)] \\ &> y \cdot \sum_{k \geq K/2} P_k(x, \lambda) \cdot C + y \cdot \sum_{k < K/2} P_k(x, \lambda) \cdot \mathbb{E}[\sigma(1, k)] + y \cdot \sum_{k < K/2} P_k(x, \lambda) \cdot \mathbb{E}[\sigma(-1, K - k)] \\ &= y \cdot \sum_{k \geq K/2} P_k(x, \lambda) \cdot C + y \cdot \sum_{k < K/2} P_k(x, \lambda) \cdot C \end{aligned}$$

where the last step uses the symmetry condition $\mathbb{E}[\sigma(1, k) + \sigma(-1, K - k)] = C$ for $k < K/2$. The last expression is yC , thus $\phi_\sigma(x) > \frac{y+yC}{1+C} \geq \frac{y+yC}{1+C} = y \geq x$. This contradicts x being a fixed point. Hence there are no fixed points of ϕ_σ that satisfy both conditions in the statement of the lemma. \square

Lemma 5. *For each $\epsilon', \epsilon'' > 0$, $p \in (0, 1)$, strategy σ^* and $0 \leq \bar{\lambda} \leq 1$ with $\phi_{\sigma^*}^{\bar{\lambda}}(x) - x \geq 2\epsilon'$ for every x with $\bar{\lambda}x + (1 - \bar{\lambda})q \leq p + 2\epsilon'$ (where $\phi_{\sigma^*}^{\bar{\lambda}}$ is the inflow accuracy function with virality weight $\bar{\lambda}$), there is some N and some $\delta > 0$ so that for every σ with $\|\sigma - \sigma^*\|_2 < \delta$ and λ with $|\lambda - \bar{\lambda}| < \delta$, we have $\mathbb{P}_{\sigma, \lambda}[\lambda x(t) + (1 - \lambda)q \geq p + \epsilon'/2] > 1 - \epsilon''$ for every $t \geq N$.*

Proof. Because $\phi_\sigma^\lambda(x)$ is polynomial in λ , σ , and x , there exists $\delta > 0$ such that $\phi_\sigma^\lambda(x) - x \geq \epsilon'$ for every x with $\lambda x + (1 - \lambda)q \leq p + \epsilon'$ when $\|\sigma^* - \sigma\|_2 < \delta$ and $|\bar{\lambda} - \lambda| < \delta$. Shrinking δ if necessary, we can also assume that λ is bounded away from zero when $|\bar{\lambda} - \lambda| < \delta$.

For the remainder of the proof, fix σ and λ in these neighborhoods. We will observe at the end of the proof that the bounds we will prove are uniform in the choice of σ and λ .

Let $p' > p + \epsilon'$ be the largest number in $(0, 1)$ such that

$$\phi_\sigma(x) - x \geq \epsilon'/2 \tag{6}$$

for all x satisfying $\lambda x + (1 - \lambda)q \leq p'$. Let $N_1 < N_2$ be positive integers with $N_2 \geq bN_1$ for some integer $b > 1$. We will first show that for N_1 and N_2 large enough, the probability that $\lambda x(t) + (1 - \lambda)q < p'$ for all $t \in [N_1, N_2]$ is small. We will then show that if $\lambda x(t_1) + (1 - \lambda)q > p'$ for some $N_1 \leq t_1 < N_2$, then the probability that $\lambda x(N_2) + (1 - \lambda)q < p + \epsilon'/2$ is small.

By the Chernoff bound applied to $z(t)$ and compactness of the set of strategies σ under consideration, we can choose a constant $B > 0$ independent of σ such that

$$\max_{x \in [0, 1]} |\phi_{\sigma, z(t)}(x) - \phi_\sigma(x)| < \epsilon'/4 \tag{7}$$

with probability at least $1 - 2e^{-Bt}$ for t sufficiently large.

Recall that we can decompose $\mathbf{y}(t)$ as

$$\mathbf{y}(t+1) = \mathbf{y}(t) + h(\mathbf{y}(t)) + M(t+1),$$

where $h(\mathbf{y}(t))$ is deterministic and $M(t+1)$ is a martingale. We have $|M(t)| < 2(C+1)/t$ for all t . So by Theorem C.7 from Appendix C of [Borkar \(2023\)](#), for any $\alpha > 0$ and any t_1 and t_2 ,

$$\mathbb{P} \left(\sup_{t_1 < t < t_2} \left| \sum_{i=t_1}^t M(i) \right| > \alpha \right) \leq 4e^{-\frac{\alpha^2}{\sum_{i=t_1}^{t_2} \frac{1}{4(C+1)^2/i^2}}}. \quad (8)$$

Consider the event E that $\lambda x(t) + (1-\lambda)q < p'$ for all $N_1 \leq t \leq N_2$. Suppose inequality (7) holds for all $N_1 \leq t < N_2$. Then we have

$$\begin{aligned} x(N_2) - x(N_1) &= \sum_{t=N_1}^{N_2-1} \frac{\phi_{\sigma, z(t)}(x(t)) - x(t)}{t+1} + \sum_{t=N_1}^{N_2-1} M(t+1) \\ &= \sum_{t=N_1}^{N_2-1} \frac{\phi_{\sigma, z(t)}(x(t)) - \phi_{\sigma}(x(t))}{t+1} + \sum_{t=N_1}^{N_2-1} \frac{\phi_{\sigma}(x(t)) - x(t)}{t+1} + \sum_{t=N_1}^{N_2-1} M(t+1) \\ &\geq \sum_{t=N_1}^{N_2-1} \epsilon'/4 \cdot \frac{1}{t+1} + \sum_{t=N_1}^{N_2-1} M(t+1) \text{ by inequalities (6) and (7)} \\ &\geq (\epsilon'/4)(\log(N_2) - \log(N_1)) + \sum_{t=N_1}^{N_2-1} M(t+1). \end{aligned}$$

When event E holds, the right-hand side must be at most p'/λ . Taking b and therefore N_2/N_1 sufficiently large, we can assume that

$$(\epsilon'/4)(\log(N_2) - \log(N_1)) > 2p'/\lambda.$$

By equation (8), the absolute value of the sum of martingales is greater than p'/λ with probability at most

$$4e^{-\frac{(p'/\lambda)^2}{\sum_{i=N_1}^{N_2} \frac{1}{4(C+1)^2/(i+1)^2}}} \leq 4e^{-\frac{(p'/\lambda)^2 N_1 N_2}{8(C+1)^2(N_2-N_1)}} < 4e^{-\frac{(p'/\lambda)^2 N_1}{8(C+1)^2}}.$$

Along with the Chernoff bound, this gives an upper bound on the probability of event E .

If event E does not hold, there exists some $N_1 \leq t \leq N_2$ such that $\lambda x(t) + (1-\lambda)q \geq p'$. Choose t_1 so that $t_1 - 1$ is the largest such t .

Suppose $\lambda x(N_2) + (1-\lambda)q \leq p + \epsilon'/2$. For N_1 sufficiently large, this implies $t_1 \leq N_2$. So we must have

$$x(N_2) - x(t_1) \leq ((p + \epsilon'/2) - p')/\lambda < -\epsilon'/(2\lambda).$$

On the other hand, when inequality (7) holds for all $N_1 \leq t < N_2$,

$$\begin{aligned}
x(N_2) - x(t_1) &= \sum_{t=t_1}^{N_2-1} \frac{\phi_{\sigma, z(t)}(x(t)) - x(t)}{t+1} + \sum_{t=t_1}^{N_2-1} M(t+1) \\
&= \sum_{t=t_1}^{N_2-1} \frac{\phi_{\sigma, z(t)}(x(t)) - \phi_{\sigma}(x(t))}{t+1} + \sum_{t=t_1}^{N_2-1} \frac{\phi_{\sigma}(x(t)) - x(t)}{t+1} + \sum_{t=t_1}^{N_2-1} M(t+1) \\
&\geq \sum_{t=t_1}^{N_2-1} \epsilon'/4 \cdot \frac{1}{t+1} + \sum_{t=t_1}^{N_2-1} M(t+1) \text{ by inequalities (6) and (7)} \\
&\geq (\epsilon'/4)(\log(N_2) - \log(t_1)) + \sum_{t=t_1}^{N_2-1} M(t+1).
\end{aligned}$$

Applying equation (8) with $\alpha = \epsilon'/(2\lambda)$, the absolute value of the sum of martingales is greater than $\epsilon'/(2\lambda)$ with probability at most

$$4e^{-\frac{(\epsilon')^2/(2\lambda)^2}{\sum_{i=t_1}^{N_2} 4(C+1)^{2/(i+1)^2}}} \leq 4e^{-\frac{(\epsilon'/\lambda)^2 N_2 t_1}{32(C+1)^2(N_2-t_1)}} \leq 4e^{-\frac{(\epsilon'/\lambda)^2 t_1}{32(C+1)^2}}.$$

When this does not hold and the Chernoff bounds apply, $x(N_2) - x(t_1)$ is greater than $-\epsilon'/(2\lambda)$ and therefore $\lambda x(N_2) + (1-\lambda)q > p + \epsilon'/2$ if N_1 is sufficiently large. This gives an upper bound on the probability that $\lambda x(N_2) + (1-\lambda)q \leq p + \epsilon'/2$.

We conclude that

$$\mathbb{P}_{\sigma}[\lambda x(N_2) + (1-\lambda)q < p + \epsilon'/2] \leq 4e^{-\frac{(p'/\lambda)^2 N_1}{8(C+1)^2}} + \sum_{t=N_1+1}^{N_2-1} 4e^{-\frac{(\epsilon'/\lambda)^2 t}{32(C+1)^2}} + 2 \sum_{t=N_1}^{N_2-1} e^{-Bt}$$

for N_1 sufficiently large. Because the second and third terms are geometric series, we can choose N_1 sufficiently large so that this probability is less than ϵ'' for all $N_2 \geq bN_1$. Because λ is bounded away from zero, we can make this choice uniformly in λ and σ (subject to the constraints $|\lambda - \bar{\lambda}| < \delta$ and $\|\sigma - \sigma^*\|_2 < \delta$). So for N_1 sufficiently large, we have

$$\mathbb{P}_{\sigma}[\lambda x(t) + (1-\lambda)q \geq p + \epsilon'/2] > 1 - \epsilon''$$

for $t \geq N = bN_1$. □

A.5.2 Proof of Theorem 2

Proof. **Part 1:** Fix $0 < \lambda \leq \lambda^*$ and suppose σ^* is a limit equilibrium.

Step 1: Either $\sigma^* = \sigma^{\text{maj}}$, or all fixed points of ϕ_{σ^*} are strictly informative.

We verify that σ^* satisfies the hypotheses of Lemma 3. Note σ^* is the limit of a sequence of symmetric BNEs $(\sigma^{(i)})$, where every $\sigma^{(i)}$ is state symmetric. Also, in the i -th finite society under the equilibrium $\sigma^{(i)}$, belief about $\{\omega = 1\}$ must be weakly higher after observing k positive stories and $s = 1$ than k positive stories and $s = -1$ for every $0 \leq k \leq K$. So by optimality of $\sigma^{(i)}$, we have $\mathbb{E}[\sigma^{(i)}(1, k)] \geq \mathbb{E}[\sigma^{(i)}(-1, k)]$ for every i and every $0 \leq k \leq K$. The limit σ^* must also satisfy state symmetry and $\mathbb{E}[\sigma^*(1, k)] \geq \mathbb{E}[\sigma^*(-1, k)]$ for every $0 \leq k \leq K$. Also, when K is even, by the state symmetry of the equilibrium $\sigma^{(i)}$ we know that a news feed with $K/2$ positive stories in society i generates an equilibrium posterior belief that both states are equally likely. Thus, optimality of $\sigma^{(i)}$ implies $\sigma^{(i)}(1, K/2)(C) = 1$ and $\sigma^{(i)}(-1, K/2)(0) = 1$. The limit σ^* must then also satisfy $\sigma^*(1, K/2)(C) = 1$, $\sigma^*(-1, K/2)(0) = 1$.

If ϕ_{σ^*} has a strictly misleading fixed point and $\sigma^* \neq \sigma^{\text{maj}}$, that is some $x \in [0, 1]$ with $\lambda x + (1 - \lambda)q < 1/2$ and such that $\phi_{\sigma^*}(x) = x$, then by Lemma 3 we get $\phi_{\sigma^{\text{maj}}}(x) < x$. But we also have $\phi_{\sigma^{\text{maj}}}(0) > 0$, which means $\phi_{\sigma^{\text{maj}}}$ has a strictly misleading fixed point in $(0, x)$ by the intermediate-value theorem, and further $\phi_{\sigma^{\text{maj}}}$ will continue to have a nearby fixed point for nearby values of λ . Since $x \leq 1/2$, this implies for some $\lambda' < \lambda^*$, $\phi_{\sigma^{\text{maj}}}$ has a fixed point in $[0, 1/2]$, which contradicts the definition of λ^* .

If there is some x with $\lambda x + (1 - \lambda)q = 1/2$ and such that $\phi_{\sigma^*}(x) = x$, then by Lemma 3 we get $\phi_{\sigma^{\text{maj}}}(x) \leq x$. But since the sampling accuracy at x is exactly $1/2$, every sample is as likely as its mirror image, so the majority rule is expected to share at least $C/2$ correct stories out of C , hence $\phi_{\sigma^{\text{maj}}}(x) > 1/2$ after accounting for the arrival of new stories that tend to match the true state. This is a contradiction. Thus, every fixed point of ϕ_{σ^*} must be strictly informative unless $\sigma^* = \sigma^{\text{maj}}$.

Step 2: If $\lambda < \lambda^*$, ϕ_{σ^*} only has fixed points in $(q, 1]$. If $\lambda = \lambda^*$, either $\sigma^* = \sigma^{\text{maj}}$ or ϕ_{σ^*} only has fixed points in $(q, 1]$.

We first show all fixed points of ϕ_{σ^*} are strictly informative, except when $\lambda = \lambda^*$ and $\sigma^* = \sigma^{\text{maj}}$. If $\lambda < \lambda^*$ and $\sigma^* = \sigma^{\text{maj}}$, by definition of λ^* we know that all fixed points of ϕ_{σ^*} are strictly

informative. And if $\sigma^* \neq \sigma^{\text{maj}}$, then by **Step 1**, all fixed points of ϕ_{σ^*} are strictly informative. If $\lambda = \lambda^*$ and $\sigma^* \neq \sigma^{\text{maj}}$, again **Step 1** implies all fixed points of ϕ_{σ^*} are strictly informative.

We verify that, except when $\lambda = \lambda^*$ and $\sigma^* = \sigma^{\text{maj}}$, σ^* is such that $\sigma^*(1, k)(C) = 1$ for every $k \geq K/2$, and thus satisfies the hypotheses of Lemma 4. Since all fixed points of ϕ_{σ^*} are strictly informative, there exists some $\epsilon' > 0$ so that $\phi_{\sigma^*}(x) - x > 2\epsilon'$ for every x where $\lambda x + (1 - \lambda)q \leq 0.5 + 2\epsilon'$. Find some $\epsilon'' > 0$ so that

$$\frac{(0.5 + \epsilon'/4)^{\lfloor K/2 \rfloor + 1} \cdot (0.5 - \epsilon'/4)^{K - \lfloor K/2 \rfloor - 1} \cdot (1 - 3\epsilon'')}{(0.5 + \epsilon'/4)^{K - \lfloor K/2 \rfloor - 1} \cdot (0.5 - \epsilon'/4)^{\lfloor K/2 \rfloor + 1} \cdot (1 - 3\epsilon'') + 3\epsilon''} > 1. \quad (9)$$

Apply Lemma 5 to these ϵ', ϵ'' and $p = 1/2$ to find N and δ . Also, by the law of large numbers, we may find N' so that $\mathbb{P}[|\text{Binom}(t, q)/t - q| > \epsilon'/4] < \epsilon''$ whenever $t \geq N'$, where $\text{Binom}(t, q)$ refers to a binomial random variable with t trials and q success probability. Since $\sigma^{(i)} \rightarrow \sigma^*$, there exists I so that $\|\sigma^{(i)} - \sigma^*\|_2 < \delta$ whenever $i \geq I$. For all $i \geq I$ large enough, we must have $\max(N, N')/(n_i - K) < \epsilon''$. But in the equilibrium $\sigma^{(i)}$ in society i , we know that agents' equilibrium belief about sampling accuracy for any position in $\{\max(N, N'), \max(N, N') + 1, \dots, n_i\}$ assigns mass at least $1 - 2\epsilon''$ to the region $[0.5 + \epsilon'/4, 1]$. This is because if we have both $\lambda x(t) + (1 - \lambda)q \geq 0.5 + \epsilon'/2$ and $|z(t) - q| < \epsilon'/4$, then we also have $\lambda x(t) + (1 - \lambda)z(t) \geq 0.5 + \epsilon'/4$ — the former event has probability at least $1 - \epsilon''$ when $t \geq N$ and latter event has probability at least $1 - \epsilon''$ when $t \geq N'$. Hence, equilibrium belief about sampling accuracy for a uniformly random position in $\{K + 1, K + 2, \dots, \max(N, N'), \max(N, N') + 1, \dots, n_i\}$ assigns mass at least $1 - 3\epsilon''$ to the same region. Thus, the expression on the LHS of Equation (9) gives a lower bound on the the posterior likelihood ratio of $\omega = 1$ and $\omega = -1$ after seeing a sample containing k positive stories in equilibrium $\sigma^{(i)}$, for any $k > K/2$. Hence, by optimality, $\sigma^{(i)}(1, k)(C) = 1$ for every $k > K/2$. Also, for any belief about sampling accuracy, a sample with $k = K/2$ is uninformative, so if $K/2$ is an integer then $\sigma^{(i)}(1, K/2)(C) = 1$ by optimality. Thus we see for all large enough j , $\sigma^{(j)}(1, k)(C) = 1$ for every $k \geq K/2$, hence the same must hold for the limit σ^* .

Combining the conclusion of Lemma 4 (which rules out steady states at or lower than q with a sampling accuracy strictly higher than $1/2$) with the argument at the beginning of **Step 2** (which rules out steady states with sampling accuracy $1/2$ or lower), we have completed this step.

Step 3: $\sigma^* = \sigma^{\text{maj}}$.

By **Step 2**, we just need to establish this when ϕ_{σ^*} only has fixed points in $(q, 1]$. By state symmetry it suffices to show that $\sigma^*(-1, k)(C) = 1$ for every $k > K/2$. Since ϕ_{σ^*} only has fixed points in $(q, 1]$, there exists some $\epsilon' > 0$ so that $\phi_{\sigma^*}(x) - x \geq 2\epsilon'$ for every x where $\lambda x + (1 - \lambda)q \leq q + 2\epsilon'$. Find some $\epsilon'' > 0$ so that

$$\frac{(q + \epsilon'/4)^{\lfloor K/2 \rfloor + 1} \cdot (1 - q - \epsilon'/4)^{K - \lfloor K/2 \rfloor - 1} \cdot (1 - 3\epsilon'')}{(q + \epsilon'/4)^{K - \lfloor K/2 \rfloor - 1} \cdot (1 - q - \epsilon'/4)^{\lfloor K/2 \rfloor + 1} \cdot (1 - 3\epsilon'') + 3\epsilon''} > \frac{q}{1 - q}. \quad (10)$$

Apply Lemma 5 to these ϵ', ϵ'' and $p = q$ to find N and δ . Also, by the law of large numbers, we may find N' so that $\mathbb{P}[|\text{Binom}(t, q)/t - q| > \epsilon'/4] < \epsilon''$ whenever $t \geq N'$, where $\text{Binom}(t, q)$ refers to a binomial random variable with t trials and q success probability. Since $\sigma^{(i)} \rightarrow \sigma^*$, there exists I so that $\|\sigma^{(i)} - \sigma^*\|_2 < \delta$ whenever $i \geq I$. For all $i \geq I$ large enough, by the same arguments as in **Step 2**, the expression on the LHS of Equation (10) gives a lower bound on the the posterior likelihood ratio of $\omega = 1$ and $\omega = -1$ after seeing a sample containing k positive stories in equilibrium $\sigma^{(i)}$ in society i , for any $k > K/2$. So even if the private signal is $s = -1$, the $\omega = 1$ state is still strictly more likely, and $\sigma^{(i)}(-1, k)(C) = 1$ by optimality. So we also have in the limit $\sigma^*(-1, k)(C) = 1$ for every $k > K/2$.

Part 2: For $\lambda < \lambda^*$, σ^{maj} has a unique steady state.

For K odd, we have

$$\phi_{\sigma^{\text{maj}}}(x) = \frac{q + C \sum_{k > K/2} P_k(x, \lambda)}{1 + C}.$$

By straightforward algebra, we can show that

$$\phi'_{\sigma^{\text{maj}}}(x) = \frac{C\lambda K}{1 + C} \mathbb{P} \left[\text{Binom}(K - 1, \lambda x + (1 - \lambda)q) = \frac{K - 1}{2} \right].$$

If $\lambda = 0$, then $\phi'_{\sigma^{\text{maj}}}(x)$ is constant in x , so $\phi_{\sigma^{\text{maj}}}(x)$ cannot intersect $y = x$ multiple times. If $\lambda > 0$, then $\phi'_{\sigma^{\text{maj}}}(x)$ is strictly decreasing in $x \in (\frac{1}{2}, 1]$, so $\phi_{\sigma^{\text{maj}}}(x)$ is strictly concave for $x \in (\frac{1}{2}, 1]$. Since $\phi_{\sigma^{\text{maj}}}(0) > 0$, at the first fixed point x^* with $\phi_{\sigma^{\text{maj}}}(x^*) = x^*$, we must have $\phi'_{\sigma^{\text{maj}}}(x^*) \leq 1$. We have $x^* > 1/2$ by **Part 1**, so strict concavity of $\phi_{\sigma^{\text{maj}}}(x)$ in $(\frac{1}{2}, 1]$ ensures that there are no fixed points larger than x^* . There is a unique informative steady state.

For K even, we have

$$\phi_{\sigma^{\text{maj}}}(x) = \frac{q + qC \sum_{k \geq K/2} P(k, \lambda x + (1 - \lambda)q) + (1 - q)C \sum_{k \geq (K/2)+1} P(k, \lambda x + (1 - \lambda)q)}{1 + C}.$$

So

$$\phi'_{\sigma^{\text{maj}}}(x) = \frac{C\lambda K}{1 + C} \cdot \left(q\mathbb{P} \left[\text{Binom}(K - 1, y) = \frac{K}{2} - 1 \right] + (1 - q)\mathbb{P} \left[\text{Binom}(K - 1, y) = \frac{K}{2} \right] \right),$$

where $y = \lambda x + (1 - \lambda)q$. The term in parentheses is proportional to

$$g(y) = (q(1 - y) + (1 - q)y)(y(1 - y))^{K/2-1}.$$

For $\lambda > 0$, the derivative $\frac{\partial g}{\partial x}$ has the same sign as the derivative

$$\frac{\partial g}{\partial y} = (y(1 - y))^{K/2-2}((K/2 - 1)(2y - 1)(q(2y - 1) - y) - (2q - 1)(1 - y)y),$$

which is strictly negative for $y \in (\frac{1}{2}, 1)$.

Since $y \in (\frac{1}{2}, 1)$ whenever $x \in (\frac{1}{2}, 1)$, we conclude that $\phi_{\sigma^{\text{maj}}}(x)$ is strictly concave for $x \in (\frac{1}{2}, 1)$ if $\lambda > 0$. By the same arguments as before, there is a unique informative steady state.

Part 3: Now, suppose $\lambda > \lambda^*$ and suppose σ^* is a limit equilibrium.

Step 1: ϕ_{σ^*} must have a weakly misleading fixed point.

If not, then there exists some $\epsilon > 0$ so that $\phi_{\sigma^*}(x) - x > \epsilon$ for every x where $\lambda x + (1 - \lambda)q \leq 0.5 + \epsilon$.

By repeating the arguments in Part 1, Steps 2 and 3, we conclude $\sigma^* = \sigma^{\text{maj}}$.

But we show σ^{maj} has a strictly misleading fixed point for every $\lambda > \lambda^*$. By the definition of λ^* , we can choose some λ' with $\lambda^* \leq \lambda' < \lambda$ such that there exists a strictly misleading fixed point x' under σ^{maj} at λ' (we get “strictly” because by Lemma 1, $1/2$ is not a fixed point of σ^{maj} and all fixed points in $[0, 1/2)$ are strictly misleading). We rewrite the inflow accuracy function $\phi_{\sigma}(x)$ as $\phi_{\sigma}(x, \lambda)$ to make explicit its dependence on λ .

Observe $\phi_{\sigma}(x, \lambda)$ only depends on x and λ through the value of $\lambda x + (1 - \lambda)q$. We can define x by

$$\lambda x + (1 - \lambda)q = \lambda' x' + (1 - \lambda')q.$$

Since $\lambda' < \lambda$ and $x' < q$, this equality implies that $x > x'$. For x' to be a strictly misleading fixed point under the majority rule we must have $\lambda'x' + (1 - \lambda')q < \frac{1}{2}$, and therefore $x < \frac{1}{2}$ as well.

So

$$\phi_{\sigma^{\text{maj}}}(x, \lambda) = \phi_{\sigma^{\text{maj}}}(x', \lambda') = x',$$

where the second inequality holds because x' is a fixed point under σ^{maj} and λ' . So we conclude that $\phi_{\sigma^{\text{maj}}}(x, \lambda) < x$. Since $\phi_{\sigma^{\text{maj}}}(-1, \lambda) > 0$, by the intermediate value theorem there is some fixed point of $\phi_{\sigma^{\text{maj}}}$ between 0 and x . Since $x < \frac{1}{2}$, this is a strictly misleading fixed point, contradiction.

Note that since $\phi_{\sigma^*}(0) > 0$, the first weakly misleading fixed point of ϕ_{σ^*} is stable at least from the left, so it is also a weakly misleading steady state.

Step 2: ϕ_{σ^*} cannot have a fixed point with a sampling accuracy of exactly $1/2$.

Each $\sigma^{(i)}$, by optimality, has the property that $\mathbb{E}[\sigma^{(i)}(1, k)] \geq \mathbb{E}[\sigma^{(i)}(-1, k)]$ for every $0 \leq k \leq K$. So we must have $\mathbb{E}[\sigma^*(1, k)] \geq \mathbb{E}[\sigma^*(-1, k)]$ for each $0 \leq k \leq K$. Suppose $\lambda x + (1 - \lambda)q = 1/2$ and $\phi_{\sigma^*}(x) = x$. For each $0 \leq k < K/2$, we get

$$\begin{aligned} & P_k(x, \lambda) \cdot [q \cdot \mathbb{E}[\sigma^*(1, k)] + (1 - q) \cdot \mathbb{E}[\sigma^*(-1, k)]] + P_{K-k}(x, \lambda) \cdot [q \cdot \mathbb{E}[\sigma^*(1, K - k)] + (1 - q) \cdot \mathbb{E}[\sigma^*(-1, K - k)]] \\ &= P_k(x, \lambda) \cdot [q \cdot \mathbb{E}[\sigma^*(1, k)] + (1 - q) \cdot \mathbb{E}[\sigma^*(-1, k)]] + q \cdot \mathbb{E}[\sigma^*(1, K - k)] + (1 - q) \cdot \mathbb{E}[\sigma^*(-1, K - k)] \\ &\quad \text{since } P_k(x, \lambda) = P_{K-k}(x, \lambda) \\ &= P_k(x, \lambda) \cdot [q \cdot \mathbb{E}[\sigma^*(1, k)] + (1 - q) \cdot \mathbb{E}[\sigma^*(-1, k)]] + q \cdot (C - \mathbb{E}[\sigma^*(-1, k)]) + (1 - q) \cdot (C - \mathbb{E}[\sigma^*(1, k)]) \\ &= P_k(x, \lambda) \cdot [C + (2q - 1) \cdot \mathbb{E}[\sigma^*(1, k)] - (2q - 1) \mathbb{E}[\sigma^*(-1, k)]] \\ &\geq P_k(x, \lambda) \cdot C \text{ because } \mathbb{E}[\sigma^*(1, k)] \geq \mathbb{E}[\sigma^*(-1, k)] \text{ and } 2q - 1 > 0 \\ &\geq P_k(x, \lambda) \cdot [C/2] + P_{K-k}(x, \lambda) \cdot [C/2] \end{aligned}$$

Also, if $K/2$ is an integer, we have $\mathbb{E}[\sigma^*(1, K/2)] + \mathbb{E}[\sigma^*(-1, K/2)] = C$ and $\mathbb{E}[\sigma^*(1, K/2)] \geq \mathbb{E}[\sigma^*(-1, K/2)]$, so $q > 1 - q$ implies $q \cdot \mathbb{E}[\sigma^*(1, K/2)] + (1 - q) \cdot \mathbb{E}[\sigma^*(-1, K/2)] \geq C/2$. Thus, we conclude $\sum_{k=0}^K P_k(x, \lambda) \cdot [q \cdot \mathbb{E}[\sigma^*(1, k)] + (1 - q) \cdot \mathbb{E}[\sigma^*(-1, k)]] \geq C/2$. So

$$\phi_{\sigma^*}(x) := \frac{q + \sum_{k=0}^K P_k(x, \lambda) \cdot [q \cdot \mathbb{E}[\sigma^*(1, k)] + (1 - q) \cdot \mathbb{E}[\sigma^*(-1, k)]]}{1 + C} \geq \frac{q + C/2}{1 + C} > 1/2$$

since $q > 1/2$. But this means $\lambda\phi_{\sigma^*}(x) + (1 - \lambda)q > 1/2$, contradiction. \square

A.6 Proof of Proposition 3

Proof. Let $\lambda < \lambda' < \lambda^*$ and suppose that x^* is a steady state under λ . We want to show that there exists a steady state $(x')^* > x^*$ under λ' .

As in the proof of Part 3 of Theorem 2, let $\phi_\sigma(x, \lambda)$ be the inflow accuracy function with its dependence on λ . By monotonicity, $\phi_{\sigma^{\text{maj}}}(x, \lambda)$ is strictly increasing in λ when $x > q$. By Theorem 2, we have $x^* > q$ and therefore

$$x^* = \phi_{\sigma^{\text{maj}}}(x^*, \lambda) < \phi_{\sigma^{\text{maj}}}(x^*, \lambda').$$

Since $\phi_{\sigma^{\text{maj}}}(1, \lambda') < 1$, by the intermediate value theorem there exists $(x')^* \in (x^*, 1)$ such that

$$\phi_{\sigma^{\text{maj}}}((x')^*, \lambda') = (x')^*.$$

This is a steady state under λ' that is greater than x^* .

□

A.7 Proof of Proposition 4

Proof. Fix some K , and let $\lambda^*(q, C)$ be the critical virality value for (K, C, q) . Write $\phi_{\sigma^{\text{maj}}}(x; \lambda, q, C)$ for the inflow accuracy function, emphasizing its dependence on the parameters.

First, note that if $\lambda^*(q, C)$ is finite, then at every $\lambda \geq \lambda^*(q, C)$, $\phi_{\sigma^{\text{maj}}}(x; \lambda, q, C)$ has a strictly misleading steady state.

Part 1: $\lambda^*(q, C)$ increases when q increases. We have $q \mapsto \phi_{\sigma^{\text{maj}}}(x; \lambda, q, C)$ is always strictly increasing. For K odd, we may write $\phi_{\sigma^{\text{maj}}}(x; \lambda, q, C) = \frac{q + \sum_{k=(K+1)/2}^K P_k(x, \lambda) \cdot C}{1 + C}$. For any $p' > p$, we can think of the experiment of tossing K coins each with a probability p' of landing heads as the experiment of first tossing K coins each with a probability p of landing heads, and then independently flipping each tail to a head with some probability h so that $p + (1 - p)h = p'$. This shows that $\sum_{k=(K+1)/2}^K P_k(x, \lambda)$ is strictly increasing in $\lambda x + (1 - \lambda)q$ (since the possibility of changing some tails to heads can only increase the total number of heads in the experiment), so it is weakly increasing in q . Also, the numerator of $\phi_{\sigma^{\text{maj}}}(x; \lambda, q, C)$ contains the term q , so this shows the entire numerator is strictly increasing in q . For K even, we may write $\phi_{\sigma^{\text{maj}}}(x; \lambda, q, C) =$

$\frac{q + [qP_{K/2}(x, \lambda) + \sum_{k=(K/2)+1}^K P_k(x, \lambda)] \cdot C}{1+C}$. For any fixed $\bar{q} < 1$, $\bar{q}P_{K/2}(x, \lambda) + \sum_{k=(K/2)+1}^K P_k(x, \lambda)$ is strictly increasing in $\lambda x + (1 - \lambda)q$. This is because the possibility of changing some heads to tails keeps outcomes where $k \geq (K/2) + 1$ in this same class, while outcomes with $k = K/2$ have a positive probability of changing to the class $k \geq (K/2) + 1$, thus contribution 1 instead of $\bar{q} < 1$ to the sum. Hence $qP_{K/2}(x, \lambda) + \sum_{k=(K/2)+1}^K P_k(x, \lambda)$ is weakly increasing in q , and the numerator of $\phi_{\sigma^{\text{maj}}}(x; \lambda, q, C)$ is strictly increasing in q .

Suppose $\lambda^*(q, C) = \infty$ and $\phi_{\sigma^{\text{maj}}}(x; \lambda, q, C)$ has no root in $x \in [0, 1/2]$ for any $\lambda \in [0, 1]$. Since $\phi_{\sigma^{\text{maj}}}(0; \lambda, q, C) > 0$, by continuity this means for every $\lambda \in [0, 1]$, $\phi_{\sigma^{\text{maj}}}(x; \lambda, q, C) > x$ for each $x \in [0, 1/2]$. For any $q' > q$, we have $\phi_{\sigma^{\text{maj}}}(x; \lambda, q', C) > \phi_{\sigma^{\text{maj}}}(x; \lambda, q, C) > x$ for every $x \in [0, 1/2]$ and $\lambda \in [0, 1]$. So again, $\lambda^*(q', C) = \infty$.

Now suppose $\lambda^*(q, C)$ is finite. We know also that $\lambda^*(q, C) > 0$ since $\phi_{\sigma^{\text{maj}}}(x; \lambda, q, C)$ has no fixed point in $x \in [0, 1/2]$ when λ is near enough 0. If we have $\phi_{\sigma^{\text{maj}}}(x'; \lambda^*(q, C), q, C) < x'$ for any $x' \in [0, 1/2]$, then by continuity there is some $0 < \lambda < \lambda^*(q, C)$ that still has $\phi_{\sigma^{\text{maj}}}(x'; \lambda, q, C) < x'$, which means $\phi_{\sigma^{\text{maj}}}(x; \lambda, q, C)$ has a root in $x \in [0, 1/2)$ by the intermediate-value theorem. This contradicts the definition of $\lambda^*(q, C)$. So we must instead have $\phi_{\sigma^{\text{maj}}}(x; \lambda^*(q, C), q, C) \geq x$ for every $x \in [0, 1/2]$. This means for every $q' > q$, $\phi_{\sigma^{\text{maj}}}(x; \lambda^*(q, C), q', C) > x$ for every $x \in [0, 1/2]$, that is $\phi_{\sigma^{\text{maj}}}(x; \lambda^*(q, C), q', C)$ has no fixed point in $[0, 1/2]$. This means that $\phi_{\sigma^{\text{maj}}}(x; \lambda^*(q, C), q', C)$ does not have a strictly misleading steady state. So either $\lambda^*(q', C) = \infty$, or $\lambda^*(q', C)$ is finite but $\lambda^*(q', C) > \lambda^*(q, C)$.

Part 2: $\lambda^*(q, C)$ increases when C decreases. If $C' < C$, then $\phi_{\sigma^{\text{maj}}}(x; \lambda, q, C') > \phi_{\sigma^{\text{maj}}}(x; \lambda, q, C)$ at every x where $\lambda x + (1 - \lambda)q \leq 1/2$. To see this, first suppose K is odd. Then at such x , $\sum_{k=(K+1)/2}^K P_k(x, \lambda) \leq 1/2$, and we have

$$\begin{aligned} \frac{d}{dC} \frac{q + \sum_{k=(K+1)/2}^K P_k(x, \lambda) \cdot C}{1+C} &= \frac{\sum_{k=(K+1)/2}^K P_k(x, \lambda) \cdot (1+C) - (q + \sum_{k=(K+1)/2}^K P_k(x, \lambda) \cdot C)}{(1+C)^2} \\ &= \frac{\sum_{k=(K+1)/2}^K P_k(x, \lambda) - q}{(1+C)^2} < 0 \end{aligned}$$

using the fact that $q > 1/2$.

If instead K is even, then we have $P_k(x, \lambda) \leq P_{K-k}(x, \lambda)$ for every $k \geq (K/2) + 1$. This means $\sum_{k=(K/2)+1}^K P_k(x, \lambda) \leq \frac{1}{2} \cdot [1 - P_{K/2}(x, \lambda)]$, so then $qP_{K/2}(x, \lambda) + \sum_{k=(K/2)+1}^K P_k(x, \lambda) < q$ since

$q > 1/2$. We have:

$$\begin{aligned}
& \frac{d}{dC} \left(\frac{q + [qP_{K/2}(x, \lambda) + \sum_{k=(K/2)+1}^K P_k(x, \lambda)] \cdot C}{1 + C} \right) \\
&= \frac{[qP_{K/2}(x, \lambda) + \sum_{k=(K/2)+1}^K P_k(x, \lambda)] \cdot (1 + C) - (q + [qP_{K/2}(x, \lambda) + \sum_{k=(K/2)+1}^K P_k(x, \lambda)] \cdot C)}{(1 + C)^2} \\
&= \frac{qP_{K/2}(x, \lambda) + \sum_{k=(K/2)+1}^K P_k(x, \lambda) - q}{(1 + C)^2} < 0
\end{aligned}$$

Suppose $\lambda^*(q, C) = \infty$ and $\phi_{\sigma^{\text{maj}}}(x; \lambda, q, C)$ has no root in $x \in [0, 1/2]$ for any $\lambda \in [0, 1]$. Since $\phi_{\sigma^{\text{maj}}}(0; \lambda, q, C) > 0$, by continuity this means for every $\lambda \in [0, 1]$, $\phi_{\sigma^{\text{maj}}}(x; \lambda, q, C) > x$ for each x with $\lambda x + (1 - \lambda)q \leq 1/2$. For any $C' < C$, we have $\phi_{\sigma^{\text{maj}}}(x; \lambda, q, C') > \phi_{\sigma^{\text{maj}}}(x; \lambda, q, C) > x$ for every x with $\lambda x + (1 - \lambda)q \leq 1/2$ and $\lambda \in [0, 1]$. That is, $\phi_{\sigma^{\text{maj}}}(x; \lambda, q, C')$ does not have a strictly misleading fixed point for any $\lambda \in [0, 1]$, which means $\lambda^*(q, C') = \infty$.

Now suppose $\lambda^*(q, C)$ is finite. Again, we have $\lambda^*(q, C) > 0$ and $\phi_{\sigma^{\text{maj}}}(x; \lambda^*(q, C), q, C) \geq x$ for every $x \in [0, 1/2]$ by similar arguments as before. This means for every $C' < C$, $\phi_{\sigma^{\text{maj}}}(x; \lambda^*(q, C), q, C') > x$ for every x with $\lambda x + (1 - \lambda)q \leq 1/2$, that is $\phi_{\sigma^{\text{maj}}}(x; \lambda^*(q, C), q, C')$ has no strictly misleading steady state. So either $\lambda^*(q, C') = \infty$, or $\lambda^*(q, C')$ is finite but $\lambda^*(q, C') > \lambda^*(q, C)$.

Part 3: Comparative statics in K . Now, fix q and C . For simplicity, denote $\phi_{\sigma^{\text{maj}}}(x; \lambda, q, K, C)$ by $\phi(x; \lambda, K)$, and let $p := \lambda x + (1 - \lambda)q$. Then, $P_k^{(K)}(x, \lambda) = \binom{K}{k} p^k (1 - p)^{K-k}$. We can rewrite:

$$P_k^{(K+1)}(x, \lambda) = p \cdot P_{k-1}^{(K)}(x, \lambda) + (1 - p)P_k^{(K)}(x, \lambda). \quad (11)$$

By the same arguments as in **Part 1** and **Part 2**, it suffices to show that for $0 < \lambda \leq 1$ and for every x such that $\lambda x + (1 - \lambda)q < 1/2$:

- If K is odd, then $\phi(x; \lambda, K + 1) > \phi(x; \lambda, K)$
- If $K + 1$ is even, then $\phi(x; \lambda, K + 1) > \phi(x; \lambda, K + 2)$
- For any K , $\phi(x; \lambda, K) > \phi(x; \lambda, K + 2)$

Case 1: K is odd (K to $K + 1$).

Note that

$$\phi(x; \lambda, K) = \frac{q + \sum_{k=\frac{K+1}{2}}^K P_k^{(K)}(x, \lambda) \cdot C}{1 + C}$$

and

$$\phi(x; \lambda, K+1) = \frac{q + q \cdot P_{\frac{K+1}{2}}^{(K+1)}(x, \lambda) \cdot C + \sum_{k=\frac{K+3}{2}}^{K+1} P_k^{(K+1)}(x, \lambda) \cdot C}{1 + C}$$

Applying Equation (11), we have

$$\begin{aligned} \sum_{k=\frac{K+3}{2}}^{K+1} P_k^{(K+1)}(x, \lambda) &= p \sum_{k=\frac{K+3}{2}}^{K+1} P_{k-1}^{(K)}(x, \lambda) + (1-p) \sum_{k=\frac{K+3}{2}}^{K+1} P_k^{(K)}(x, \lambda) \\ &= p \sum_{k=\frac{K+1}{2}}^K P_k^{(K)}(x, \lambda) + (1-p) \sum_{k=\frac{K+3}{2}}^K P_k^{(K)}(x, \lambda) \\ &= p \cdot P_{\frac{K+1}{2}}^{(K)}(x, \lambda) + \sum_{k=\frac{K+3}{2}}^K P_k^{(K)}(x, \lambda). \end{aligned}$$

Then,

$$\begin{aligned} &\phi(x; \lambda, K) - \phi(x; \lambda, K+1) \\ &= \frac{-q \cdot P_{\frac{K+1}{2}}^{(K+1)}(x, \lambda) \cdot C + \left[\sum_{k=\frac{K+1}{2}}^K P_k^{(K)}(x, \lambda) - \sum_{k=\frac{K+3}{2}}^{K+1} P_k^{(K+1)}(x, \lambda) \right] \cdot C}{1 + C} \\ &= \frac{-q \cdot P_{\frac{K+1}{2}}^{(K+1)}(x, \lambda) \cdot C + \left[P_{\frac{K+1}{2}}^{(K)}(x, \lambda) - p \cdot P_{\frac{K+1}{2}}^{(K)}(x, \lambda) \right] \cdot C}{1 + C} \\ &= \frac{-q \cdot P_{\frac{K+1}{2}}^{(K+1)}(x, \lambda) \cdot C + (1-p) \cdot P_{\frac{K+1}{2}}^{(K)}(x, \lambda) \cdot C}{1 + C} \\ &= \frac{\left(-q + \frac{1}{2}\right) \cdot P_{\frac{K+1}{2}}^{(K+1)}(x, \lambda) \cdot C}{1 + C} < 0, \quad (\text{using } (1-p) \cdot P_{\frac{K+1}{2}}^{(K)}(x, \lambda) = \frac{1}{2} P_{\frac{K+1}{2}}^{(K+1)}(x, \lambda)) \end{aligned}$$

since $q > \frac{1}{2}$.

Case 2: $K+1$ is even ($K+1$ to $K+2$).

Note that

$$\phi(x; \lambda, K+1) = \frac{q + q \cdot P_{\frac{K+1}{2}}^{(K+1)}(x, \lambda) \cdot C + \sum_{k=\frac{K+3}{2}}^{K+1} P_k^{(K+1)}(x, \lambda) \cdot C}{1 + C}$$

and

$$\phi(x; \lambda, K+2) = \frac{q + \sum_{k=\frac{K+3}{2}}^{K+2} P_k^{(K+2)}(x, \lambda) \cdot C}{1 + C}$$

As in the first case, we have

$$\sum_{k=\frac{K+3}{2}}^{K+2} P_k^{(K+2)}(x, \lambda) = p \cdot P_{\frac{K+1}{2}}^{(K+1)}(x, \lambda) + \sum_{k=\frac{K+3}{2}}^{K+1} P_k^{(K+1)}(x, \lambda).$$

Therefore,

$$\begin{aligned} & \phi(x; \lambda, K+1) - \phi(x; \lambda, K+2) \\ &= \frac{(q-p) \cdot P_{\frac{K+1}{2}}^{(K+1)}(x, \lambda) \cdot C}{1+C} > 0, \end{aligned}$$

since $q > \lambda x + (1-\lambda)q = p$ for all $x \in [0, 1/2)$ and $\lambda > 0$.

Case 3: K to $K+2$ for odd K .

Combining results from **Case 1** and **Case 2**,

$$\begin{aligned} \phi(x; \lambda, K) - \phi(x; \lambda, K+2) &= \phi(x; \lambda, K) - \phi(x; \lambda, K+1) + \phi(x; \lambda, K+1) - \phi(x; \lambda, K+2) \\ &= \frac{\left(-q + \frac{1}{2}\right) \cdot P_{\frac{K+1}{2}}^{(K+1)}(x, \lambda) \cdot C + (q-p) \cdot P_{\frac{K+1}{2}}^{(K+1)}(x, \lambda) \cdot C}{1+C} \\ &= \frac{\left(-p + \frac{1}{2}\right) \cdot P_{\frac{K+1}{2}}^{(K+1)}(x, \lambda) \cdot C}{1+C} > 0, \end{aligned}$$

if $p = \lambda x + (1-\lambda)q < \frac{1}{2}$.

Case 4: $K+1$ to $K+3$ for even $K+1$.

Combining results from **Case 2** and **Case 1**,

$$\begin{aligned} \phi(x; \lambda, K+1) - \phi(x; \lambda, K+3) &= \phi(x; \lambda, K+1) - \phi(x; \lambda, K+2) + \phi(x; \lambda, K+2) - \phi(x; \lambda, K+3) \\ &= \frac{(q-p) \cdot P_{\frac{K+1}{2}}^{(K+1)}(x, \lambda) \cdot C}{1+C} + \frac{\left(-q + \frac{1}{2}\right) \cdot P_{\frac{K+3}{2}}^{(K+3)}(x, \lambda) \cdot C}{1+C}. \end{aligned}$$

We have $P_{\frac{K+1}{2}}^{(K+1)}(x, \lambda) \geq P_{\frac{K+3}{2}}^{(K+3)}(x, \lambda)$, so this expression is weakly larger than

$$\frac{\left(-q + \frac{1}{2}\right) \cdot P_{\frac{K+3}{2}}^{(K+3)}(x, \lambda) \cdot C}{1+C} + \frac{(q-p) \cdot P_{\frac{K+3}{2}}^{(K+3)}(x, \lambda) \cdot C}{1+C} = \frac{\left(\frac{1}{2} - p\right) \cdot P_{\frac{K+3}{2}}^{(K+3)}(x, \lambda) \cdot C}{1+C} > 0$$

when $p = \lambda x + (1 - \lambda)q < 1/2$. □

A.8 Proof of Proposition 5

Proof. To show that $\lambda^*(q, K, C) > 1 - \frac{1}{2q}$ for any q, K, C , we prove that $\phi_{\sigma^{\text{maj}}}(x)$ does not have fixed points in $[0, 1/2]$ when $\lambda \leq 1 - \frac{1}{2q}$. We have $\lambda x + (1 - \lambda)q \geq (1 - \lambda)q \geq \frac{1}{2}$. This means that for K odd, $\sum_{k=(K+1)/2}^K P_k(x, \lambda) \geq 1/2$. For K even, we have $q \cdot P_{K/2}(x, \lambda) + \sum_{k=(K/2)+1}^K P_k(x, \lambda) > 1/2$. So in both cases, the numerator of $\phi_{\sigma^{\text{maj}}}(x)$ is at least $q + C/2$, which means $\phi_{\sigma^{\text{maj}}}(x) \geq \frac{q+C/2}{1+C} > 1/2$ since $q > 1/2$. This shows $\phi_{\sigma^{\text{maj}}}(x) > x$ for every $x \in [0, 1/2]$.

Next, fix any $1/2 < q < 1$ and any $\bar{\lambda} > 1 - \frac{1}{2q}$. Let $0 < x' < 1/2$ be any number such that $\bar{\lambda}x' + (1 - \bar{\lambda})q < 1/2$ (such x' exists by the bound on $\bar{\lambda}$). We find integers \underline{K} and \underline{C} so that whenever $K \geq \underline{K}$, $C \geq \underline{C}$, we get $\phi_{\sigma^{\text{maj}}}(x'; q, K, C, \bar{\lambda}) < x'$. Since $x' < 1/2$ and since $\phi_{\sigma^{\text{maj}}}(0; q, K, C, \bar{\lambda}) > 0$, we know that $\phi_{\sigma^{\text{maj}}}(\cdot; q, K, C, \bar{\lambda})$ has a fixed point in $(0, 1/2)$ by the intermediate-value theorem. This implies $\lambda^*(q, C, K) \leq \bar{\lambda}$.

To construct \underline{K} and \underline{C} , let $\epsilon = x'/2$. By the law of large numbers, we can find \underline{K} so that whenever $K \geq \underline{K}$, the probability that a binomial distribution with K trials and success probability $\bar{\lambda}x' + (1 - \bar{\lambda})q < 1/2$ has strictly fewer than $K/2$ successes is larger than $1 - \epsilon$. Thus, whenever $K \geq \underline{K}$, $\phi_{\sigma^{\text{maj}}}(x'; q, K, C, \bar{\lambda}) \leq \frac{q+\epsilon C}{1+C}$. Now, increasing \underline{K} further if necessary, we can find \underline{C} large enough so that for all $C \geq \underline{C}$, we have $\frac{q+\epsilon C}{1+C} < 2\epsilon$. Whenever $K \geq \underline{K}$ and $C \geq \underline{C}$, we have $\phi_{\sigma^{\text{maj}}}(x'; q, K, C, \bar{\lambda}) < 2\epsilon = x'$ as desired. □

A.9 Proof of Proposition 6

Proof. We first show that we can choose $t_0(n)$ such that σ^{maj} is an equilibrium for n sufficiently large and $x(n) \rightarrow \bar{x}$ in probability. The main step is the following lemma.

Lemma 6. *Suppose $\lambda = 0$ for the first t_0 periods and then $\lambda = 1$ for all subsequent periods. There exists a number \bar{t} and a function $\bar{n}(t)$ so that for any $t_0 \geq \bar{t}$ and $n \geq \bar{n}(t_0)$, σ^{maj} is an equilibrium in a society with n agents. Given any $\epsilon > 0$, there exists a number \hat{t} and a function $\hat{n}(t)$ so that for any $t_0 \geq \hat{t}$ and $n \geq \hat{n}(t_0)$, we have $|x(n) - \bar{x}| < \epsilon$ under strategy σ^{maj} with probability at least $1 - \epsilon$.*

Proof. We first show the second claim. Let $\epsilon > 0$. When $\lambda = 0$ in all periods, the function $\phi_{\sigma^{\text{maj}}}(x)$ is constant with value strictly greater than q . So there is a unique steady state $\underline{x} > q$ that is

informative. We will bound $x(t_0) - \underline{x}$.

By the Chernoff bound applied to $z(t)$, for any $\delta > 0$ we can choose a constant $B > 0$ such that

$$|z(t) - q| < \delta$$

with probability at least $1 - 2e^{-Bt}$. So we can choose \hat{t}' such that this holds for all $t \geq \hat{t}'$ with probability at least $1 - \epsilon/4$. Now taking δ sufficiently small and \hat{t} sufficiently large (compared to \hat{t}'), by the law of large numbers we have $|x(t_0) - \underline{x}| < \epsilon/2$ with probability at least $1 - \epsilon/2$ for any $t_0 \geq \hat{t}$.

Now for each $t_0 \geq \hat{t}$, consider the infinite-horizon stochastic process $x(t)$ that starts with t_0 periods of $\lambda = 0$ and subsequently continued with $\lambda = 1$ and $\sigma = \sigma^{\text{maj}}$. We know $x(t)$ converges almost surely as $t \rightarrow \infty$ from Theorem 2.1 of Chapter 2 of [Borkar \(2023\)](#), which applies as in Proposition 2 because

$$\sum_{t=t_0}^{\infty} \frac{1}{t} = \infty.$$

We next show the steady state reached is \bar{x} with probability at least $1 - \epsilon$.

We can condition on the event $|x(t_0) - \underline{x}| < \epsilon/2$, which occurs with probability at least $1 - \epsilon/2$. We claim that given this event, with probability at least $1 - \epsilon/2$ there do not exist any $t_2 > t_1 > t_0$ such that $x(t_1) > \underline{x} - \epsilon/2$ and $x(t_2) < \underline{x} - \epsilon$.

We have $\phi_{\sigma^{\text{maj}}}^1(\underline{x}) > \underline{x}$ (where the superscript on ϕ denotes $\lambda = 1$) since $\underline{x} > q$. So shrinking ϵ if necessary, we can choose $\delta > 0$ so that

$$\phi_{\sigma^{\text{maj}}}^1(x) > \underline{x} + \delta \tag{12}$$

for $x \in [\underline{x} - \epsilon, \underline{x}]$. If there exist $t_2 > t_1 > t_0$ such that $x(t_1) > \underline{x} - \epsilon/2$ and $x(t_2) < \underline{x} - \epsilon$, then increasing t_1 if necessary we can assume that $x(t) \leq \underline{x}$ for all $t_1 \leq t \leq t_2$. (If necessary, increase t_0 so that we cannot have $x(t) > \underline{x}$ and $x(t+1) < \underline{x} - \epsilon$ when $t \geq t_0$.)

Applying the Chernoff bound to $z(t)$ again, we can choose a constant $B > 0$ such that

$$\max_{x \in [0,1]} |\phi_{\sigma^{\text{maj}}, z(t)}^1(x) - \phi_{\sigma^{\text{maj}}}^1(x)| < \delta \tag{13}$$

with probability at least $1 - 2e^{-Bt}$ for t sufficiently large. Increasing t_0 if necessary, we can assume the inequality (13) holds for all $t \geq t_0$ with probability at least $1 - \epsilon/4$. We also condition on this event.

As in the proof of Lemma 5, we can write $\mathbf{y}(t)$ as

$$\mathbf{y}(t+1) = \mathbf{y}(t) + h(\mathbf{y}(t)) + M(t+1),$$

where $h(\mathbf{y}(t))$ is deterministic and $M(t+1)$ is a martingale. We have $|M(t)| < 2(C+1)/t$ for all t . So by Theorem C.7 from Appendix C of Borkar (2023), for any $\alpha > 0$ and any t_1 ,

$$\mathbb{P} \left(\sup_{t_2 \in (t_1, \infty)} \left| \sum_{i=t_1}^{t_2} M(i) \right| > \alpha \right) \leq 4e^{-\frac{\alpha^2}{\sum_{i=t_1}^{\infty} \frac{1}{4(C+1)^2/i^2}}}. \quad (14)$$

We have

$$\begin{aligned} x(t_2) - x(t_1) &= \sum_{t=t_1}^{t_2-1} \frac{\phi_{\sigma, z(t)}(x(t)) - x(t)}{t+1} + \sum_{t=t_1}^{t_2-1} M(t+1) \\ &= \sum_{t=t_1}^{t_2-1} \frac{\phi_{\sigma, z(t)}(x(t)) - \phi_{\sigma}(x(t))}{t+1} + \sum_{t=t_1}^{t_2-1} \frac{\phi_{\sigma}(x(t)) - x(t)}{t+1} + \sum_{t=t_1}^{t_2-1} M(t+1) \\ &\geq - \sum_{t=t_1}^{t_2-1} \delta \cdot \frac{1}{t+1} + \sum_{t=t_1}^{t_2-1} \delta \cdot \frac{1}{t+1} + \sum_{t=t_1}^{t_2-1} M(t+1) \text{ by inequalities (12) and (13)} \\ &= \sum_{t=t_1}^{t_2-1} M(t+1). \end{aligned}$$

Recall that $x(t_2) - x(t_1) \leq -\epsilon/2$. So given t_1 , inequality (14) with $\alpha = \epsilon/2$ states that the probability that

$$x(t_2) - x(t_1) \geq \sum_{t=t_1}^{t_2-1} M(t+1)$$

for any t_2 is at most $4e^{-\frac{\epsilon^2}{\sum_{i=t_1}^{\infty} \frac{1}{16(C+1)^2/i^2}}}$. Increasing t_0 if necessary, we can assume that the sum of these probabilities over all $t_1 \geq t_0$ is at most $\epsilon/4$, proving our claim.

Combining our bounds, we conclude that $x(t) \geq \underline{x} - \epsilon$ for all $t \geq t_0$ with probability at least $1 - \epsilon$. Since \bar{x} is the only steady state in this region, we must have $x(t) \rightarrow \bar{x}$ with probability at least $1 - \epsilon$. So there is some $\hat{n}(t_0)$ so that for all $n \geq \hat{n}(t_0)$, we get $|x(n) - \bar{x}| < \epsilon/2$ under strategy

σ^{maj} with probability at least $1 - \epsilon/2$.

To complete the proof, we show the first claim that the majority rule σ^{maj} is an equilibrium when t_0 and n are sufficiently large. Note that when $x(t) = \bar{x}$, the majority rule gives a strictly higher payoff than any other pure strategy. By continuity, find $\epsilon > 0$ so that if the event $\{|x(t) - \bar{x}| < \epsilon\}$ happens with probability at least $1 - \epsilon$, then the majority rule still gives a strictly higher payoff than any other pure strategy. Using the second part of the claim just proved, find \hat{t} and $\hat{n}(t)$ so that for any $t_0 \geq \hat{t}$ and $n \geq \hat{n}(t_0)$, we have $|x(n) - \bar{x}| < \epsilon/2$ under strategy σ^{maj} with probability at least $1 - \epsilon/2$. Set $\bar{t} = \hat{t}$. For each $t_0 \geq \bar{t}$, let $\bar{n}(t_0)$ be large enough so that $\hat{n}(t_0)/\bar{n}(t_0) < \epsilon/2$. When the total number of agents is $n \geq \bar{n}(t_0)$, an agent in a uniformly random position has at least $1 - \epsilon/2$ chance of being in position $\hat{n}(t_0)$ or later, and if they are in such positions they have at least $1 - \epsilon/2$ chance of facing a current viral accuracy $x(t)$ with $|x(t) - \bar{x}| < \epsilon/2$ when all others use the strategy σ^{maj} . Thus σ^{maj} is the agent's best response. \square

We can now complete the proof that we can choose $t_0(n)$ such that σ^{maj} is an equilibrium for n sufficiently large and $x(n) \rightarrow \bar{x}$ in probability. Take any decreasing sequence $\epsilon^{(k)} \rightarrow 0$. We will construct two increasing sequences $t_0^{(k)}$ and $n^{(k)}$ inductively. Given $t_0^{(1)}, \dots, t_0^{(m)}$ and $n^{(1)}, \dots, n^{(m)}$, we can apply Lemma 6 to find numbers $t_0^{(m+1)}$ and $n^{(m+1)}$ so that for $t_0^{(m+1)}$ and for any $n \geq n^{(m+1)}$, $|x(n) - \bar{x}| < \epsilon^{(m+1)}$ under strategy σ^{maj} with probability at least $1 - \epsilon^{(m+1)}$ and σ^{maj} is an equilibrium. It is without loss to assume $t_0^{(m+1)} > \max\{t_0^{(1)}, \dots, t_0^{(m)}\}$ and $n^{(m+1)} > \max\{n^{(1)}, \dots, n^{(m)}\}$ (increasing them if necessary). Now for each n , find the largest $n^{(k)}$ so that $n \geq n^{(k)}$ and let $t_0(n) = t_0^{(k)}$ (if $n < n^{(1)}$, then set $t_0(n) = 0$). This ensures (provided $n \geq n^{(1)}$) that for this choice of $t_0(n)$, we have σ^{maj} as an equilibrium and this equilibrium induces $\mathbb{P}[|x(n) - \bar{x}| < \epsilon^{(k)}] > 1 - \epsilon^{(k)}$.

We now prove the final statement in the proposition. By Lemma 4, we have $\bar{x} > q$. Fix any virality weight λ' and state-symmetric strategy σ and suppose there is a steady state $x^* > \bar{x}$ with time-invariant virality weight λ' and strategy σ . By Theorem 1, we have $\phi_\sigma^{\lambda'}(x) = x$.

We claim that $\phi_{\sigma^{\text{maj}}}^{\lambda=\lambda'}(x^*) \geq x^*$. Recall that

$$\phi_\sigma(x) = \frac{q + \sum_{k=0}^K P_k(x, \lambda) \cdot [q \cdot \mathbb{E}[\sigma(1, k)] + (1 - q) \cdot \mathbb{E}[\sigma(-1, k)]]}{1 + C}.$$

For each $k \geq K/2$,

$$P_k(x^*, \lambda') \cdot [q \cdot \mathbb{E}[\sigma(1, k)] + (1-q) \cdot \mathbb{E}[\sigma(-1, k)]] + P_{K-k}(x^*, \lambda') \cdot [q \cdot \mathbb{E}[\sigma(1, K-k)] + (1-q) \cdot \mathbb{E}[\sigma(-1, K-k)]]$$

is maximized by setting $\sigma = \sigma^{\text{maj}}$ because σ is state-symmetric and $P_k(x^*, \lambda') > P_{K-k}(x^*, \lambda')$. This verifies the claim.

Because $x^* > q$, by monotonicity $\phi_{\sigma^{\text{maj}}}^\lambda(x^*)$ is strictly increasing in λ and therefore $\phi_{\sigma^{\text{maj}}}^{\lambda=1}(x) \geq \phi_{\sigma^{\text{maj}}}^{\lambda=\lambda'}(x)$. Combining our inequalities, $\phi_{\sigma^{\text{maj}}}^{\lambda=1}(x) > \phi_{\sigma^{\text{maj}}}^{\lambda=\lambda'}(x) = x$. Since $\phi_{\sigma^{\text{maj}}}^{\lambda=1}(1) < 1$, by the intermediate value theorem, we must have a fixed point of $\phi_{\sigma^{\text{maj}}}^{\lambda=1}$ between x^* and 1. But this contradicts the definition of \bar{x} , completing the proof. \square

A.10 Proof of Proposition 7

Proof. Suppose

$$\iota < \underline{\iota} = 1 - \max_{\lambda x + (1-\lambda)q \leq \frac{1}{2}} \frac{x}{\phi_{\sigma^{\text{maj}}}(x)}$$

and σ^* is a limit equilibrium.

By optimality, we must have $\mathbb{E}[\sigma(1, k)] \geq \mathbb{E}[\sigma(-1, k)]$ for every $0 \leq k \leq K$ and $\sigma(1, K/2)(C) = 1$, $\sigma(-1, K/2)(0) = 1$ if K is even. So Lemma 3 implies that $\phi_{\sigma^{\text{maj}}}(x) \leq \phi_{\sigma^*}(x)$ for all $x \leq \frac{1}{2}$.

If x^* is a misleading steady state under σ^* with ι fraction of bots, then we must have $\lambda x^* + (1-\lambda)q \leq \frac{1}{2}$ and

$$(1-\iota)\phi_{\sigma^*}(x^*) = x^*.$$

Since $\phi_{\sigma^{\text{maj}}}(x^*) \leq \phi_{\sigma^*}(x^*)$, this means

$$\iota \geq 1 - \frac{x^*}{\phi_{\sigma^{\text{maj}}}(x^*)}.$$

But

$$\iota < 1 - \max_{\lambda x + (1-\lambda)q \leq \frac{1}{2}} \frac{x}{\phi_{\sigma^{\text{maj}}}(x)} \leq 1 - \frac{x^*}{\phi_{\sigma^{\text{maj}}}(x^*)},$$

giving a contradiction. We conclude there is no such x^* . \square

A.11 Proof of Theorem 3

First, we establish a number of preliminary properties of the model with observable virality from Section 5. The first result (similar to Proposition 2) shows that given any state-symmetric strategy σ , the induced viral accuracy process $x(t)$ converges with probability 1.

Proposition 8. *Given a state-symmetric strategy σ , there is a finite set of steady states $X^* \subseteq (0, 1)$ such that when all agents use σ , almost surely $x(t) \rightarrow x^*$ for some $x^* \in X^*$.*

Proof. Suppose all agents use the strategy σ . Without loss of generality, condition on $\omega = 1$. Let $Y = \{\mathbf{y} = (x, v, z) \in [0, 1]^3\}$. For each t , let $\mathbf{y}(t) \in Y$ be defined so that $x(t)$ is the fraction of positive stories in the viral news pool, $v(t)$ is the size of the viral news pool divided by $t \cdot C$, and $z(t)$ is the fraction of positive stories in the regular news pool. We can write

$$\mathbf{y}(t+1) = \mathbf{y}(t) + \frac{1}{v(t) \cdot t} \cdot (\boldsymbol{\xi}(t+1) - \mathbf{y}(t)),$$

where the first coordinate of $\boldsymbol{\xi}(t+1)$ is $\left(\frac{v(t) \cdot t}{v(t) \cdot t + m} \frac{m}{C}\right) \varsigma + \left(1 - \frac{v(t) \cdot t}{v(t) \cdot t + m} \frac{m}{C}\right) x(t)$ with m being the number of the C shared regular stories in period $t+1$ that became viral and ς the fraction of these newly viral stories that are positive (when $m = 0$, the first term vanishes, and we can define $\varsigma = 1$). The second coordinate of $\boldsymbol{\xi}(t+1)$ is $v(t) + v(t) \frac{t}{t+1} \left(\frac{m}{C} - v(t)\right)$. The third coordinate of $\boldsymbol{\xi}(t+1)$ is $\left(\frac{v(t) \cdot t}{t+1}\right) \cdot \mathbf{1}\{s_{t+1} = 1\} + \left(1 - \frac{v(t) \cdot t}{t+1}\right) \cdot z(t)$.

Write $h(\mathbf{y}(t)) = \mathbb{E}[\boldsymbol{\xi}(t+1) \mid \mathbf{y}(t)] - \mathbf{y}(t)$ and $M(t+1) = \boldsymbol{\xi}(t+1) - \mathbb{E}[\boldsymbol{\xi}(t+1) \mid \mathbf{y}(t)]$. We then have $\mathbf{y}(t+1) = \mathbf{y}(t) + \frac{1}{v(t) \cdot t} \cdot (h(\mathbf{y}(t)) + M(t+1))$, and we note that Theorem 2.1 of Chapter 2 of Borkar (2009) applies to this process (see Section 2.2 of Borkar (2009)) with stochastic step sizes provided we can show:

(A1) h is Lipschitz continuous.

(A2) With probability 1, $\sum_t \frac{1}{v(t) \cdot t} = \infty$ while $\sum_t \frac{1}{(v(t) \cdot t)^2} < \infty$.

(A3) $\mathbb{E}[M(t+1) \mid \mathbf{y}(t)] = 0$ and $\{M(t)\}$ are square-integrable with $\mathbb{E}[\|M(t+1)\|^2 \mid \mathbf{y}(t)] \leq \kappa(1 + \|\mathbf{y}(t)\|^2)$ a.s. for all t and some $\kappa > 0$.

(A4) $\|\mathbf{y}(t)\|$ remains bounded a.s.

(A2) obtains by law of large numbers and (A4) is clear. For (A3), the martingale property holds by the construction of $M(t)$ and the remaining properties hold because $M(t)$ is bounded

(independent of t).

For (A1), the Lipschitz continuity of the second and third coordinates of $\mathbb{E}[\boldsymbol{\xi}(t+1) \mid \mathbf{y}(t)]$ in $\mathbf{y}(t)$ are clear. For the first coordinate, note that conditional on any $m \geq 1$, the expectation $\mathbb{E}[\varsigma \mid m]$ is the inflow accuracy function evaluated at $\mathbf{y}(t)$, which is a polynomial function (and hence Lipschitz continuous) in $x(t)$ and $z(t)$. The distribution of m , the number of shared stories that go viral in period $t+1$, does not depend on $\mathbf{y}(t)$. Therefore, the first coordinate of $\mathbb{E}[\boldsymbol{\xi}(t+1) \mid \mathbf{y}(t)]$ is also Lipschitz continuous in $\mathbf{y}(t)$.

By Theorem 2.1 of Chapter 2 of Borkar (2009), the $\mathbf{y}(t)$ process converges to a compact connected internally chain transitive invariant set of the differential equation $\dot{\mathbf{r}}(t) = h(\mathbf{r}(t))$. This internally chain transitive invariant set must be a subset of $[0, 1] \times \{\alpha\} \times \{q\}$, for the law of large numbers implies that $v(t) \rightarrow \alpha$ and $z(t) \rightarrow q$ almost surely. For the same reason as in the proof of Proposition 2, at any point \mathbf{r} in the invariant set we have $\frac{dr_1(t)}{dt} = 0$ when $\mathbf{r}(t) = \mathbf{r}$. But fixing $r_2(t) = \alpha$ and $r_3(t) = q$, the values of $r_1(t)$ such that $\frac{dr_1(t)}{dt} = 0$ are the roots of a non-linear polynomial, so there are finitely many such values. \square

In light of Proposition 8, let $\pi(\cdot \mid \sigma)$ be the distribution over steady states generated by a state-symmetric strategy σ . We define the inflow accuracy function $\phi_\sigma(x)$ to be the expected fraction of the C stories shared from the regular news feed that match the state, when current viral accuracy is x and exactly q fraction of the regular story pool is correct. So,

$$\phi_\sigma(x) := \frac{\sum_{s \in \{-1, 1\}} \sum_{k_R=0}^{K_R} \sum_{k_V=0}^{K_V} \mathbb{E}[\sigma_R(s, k_R, k_V)] \cdot \mathbb{P}[s \mid \omega = 1] \cdot \mathbb{P}[\text{Binom}(K_R, q) = k_R] \cdot \mathbb{P}[\text{Binom}(K_V, x) = k_V]}{C},$$

where $\mathbb{E}[\sigma_R(s, k_R, k_V)]$ refers to the expected number of positive regular stories shared by the mixed action $\sigma(s, k_R, k_V)$.

Since $\mathbb{P}[\text{Binom}(K_R, q) = K_R]$ and $\mathbb{P}[\text{Binom}(K_R, q) = 0]$ are both positive-probability events, the feasibility of the strategy σ implies the strict inequalities $0 < \phi_\sigma(x) < 1$ for every $x \in [0, 1]$. It is also clear that ϕ_σ is a polynomial function of its argument. The next result, analogous to Theorem 1, shows that fixed points of ϕ_σ that are stable from at least one side must be steady states.

Theorem 4. *We have $\pi(x^* \mid \sigma) > 0$ if $\phi_\sigma(x^*) = x^*$ and there exists some $\epsilon > 0$ so that either (a) $\phi_\sigma(x) < x$ for all $x \in (x^*, x^* + \epsilon)$, or (b) $\phi_\sigma(x) > x$ for all $x \in (x^* - \epsilon, x^*)$. Conversely, for $x^* \in [0, 1]$, we have $\pi(x^* \mid \sigma) > 0$ only if $\phi_\sigma(x^*) = x^*$.*

Proof. The proof is similar to the proof of Theorem 1, except we need to account for randomness in the size of the viral story pool over time and randomness in the number of shared stories that go viral in each period, and there is a different equation that defines the viral accuracy function.

We say a fixed point x^* of $\phi_\sigma(x)$ is a *touchpoint* if there exists $\epsilon > 0$ such that $\phi_\sigma(x) < x$ for all $x \neq x^*$ in $(x^* - \epsilon, x^* + \epsilon)$ or $\phi_\sigma(x) > x$ for all $x \neq x^*$ in $(x^* - \epsilon, x^* + \epsilon)$.

Case (i): x^* is a touchpoint.

The proof extends the arguments from Theorem 1 of Pemantle (1991). Suppose that $\phi_\sigma(x) > x$ for all $x \neq x^*$ in $(x^* - \epsilon, x^* + \epsilon)$. The other case is the same.

Fix $w \in (0, \frac{1}{2})$ and $w_1 \in (w, \frac{1}{2})$. Choose $\gamma > 1$ such that $\gamma w_1 < \frac{1}{2}$. Define $g(r) = re^{(1-r)/(2w_1\gamma)}$. Then $g(1) = 1$ and $g'(1) = 1 - 1/(2w_1\gamma) < 0$, so we can choose $r_0 \in (0, 1)$ with $g(r_0) > 1$. Also define

$$T(n) = e^{n(1-r_0)/(\gamma w_1)}.$$

Then

$$g(r_0)^n = r_0^n T(n)^{1/2} > 1.$$

Choose N such that $\gamma r_0^N < \epsilon$. Since $T(1)^{1/2} r_0 = g(r_0) > 1$, we can find $\chi > 0$ such that $T(1)^{1/2-\chi} > r_0$ and therefore $T(n)^{1/2-\chi} r_0^{-n} \rightarrow \infty$. Let $\tau_N = \inf\{j > T(N) : x(j-1) < x^* - r_0^N < x(j)\}$, using the convention that $\tau_N = -\infty$ if there is no such j . For each $n \geq N$, define

$$\tau_{n+1} = \inf\{j \geq \tau_n : x(j) > x^* - r_0^{n+1}\}.$$

So τ_n is the first time the stochastic process crosses $x^* - r_0^n$.

We will show the probability that $\tau_n > T(n)$ for all $n \geq N$ is positive. Since $x(t) \rightarrow x^*$ from below whenever said event holds (since $x(t)$ converges with probability 1), this will complete the case.

We first bound the probability that $z(t)$ is far from q . Define a function

$$\phi_{\sigma,z}(x) := \frac{\sum_{s \in \{-1,1\}} \sum_{k_R=0}^{K_R} \sum_{k_V=0}^{K_V} \mathbb{E}[\sigma(s, k_R, k_V)] \cdot \mathbb{P}[s \mid \omega = 1] \cdot \mathbb{P}[\text{Binom}(K_R, z) = k_R] \cdot \mathbb{P}[\text{Binom}(K_V, x) = k_V]}{C}$$

to be the inflow accuracy when a fraction z of past private signals have value 1.

We begin by defining an event \mathcal{C} under which $z(t)$ is close to q for t sufficiently large and $v(t)$ is close to α for t sufficiently large. Let \mathcal{C}_1 be the event that for all $n \geq N$ and for all $t \geq T(n)$,

$$\phi_{\sigma, z(t)}(x) - x \geq -1/T(n)^{1/2-\chi}$$

on $(x^* - \epsilon, x^* + \epsilon)$. Because $\phi_{\sigma, z}(x) - x$ is polynomial (in z and x) and is non-negative on this interval when $z = q$, this holds for $|z(t) - q| < B/T(n)^{1/2-\chi}$ for some $B > 0$.

Note that the first coordinate of $h(\mathbf{y}(t)) = \mathbb{E}[\xi(t+1) \mid \mathbf{y}(t)] - \mathbf{y}(t)$ can be written as $\mathbb{E}[(\frac{v(t) \cdot t}{v(t) \cdot t + m_{t+1}} \frac{m_{t+1}}{C}) \cdot (\varsigma_{t+1} - x(t)) \mid \mathbf{y}(t)]$, where $v(t)$ is measurable with respect to $\mathbf{y}(t)$ and m_{t+1} is independent of ς_{t+1} . Therefore we have $h_1(\mathbf{y}(t)) = \mathbb{E}[\frac{v(t) \cdot t}{v(t) \cdot t + m_{t+1}} \frac{m_{t+1}}{C} \mid v(t)] \cdot (\phi_{\sigma, z(t)}(x(t)) - x(t))$. Choose $\epsilon > 0$ small enough so that $0 < \alpha - \epsilon < \alpha + \epsilon < 1$, and that $\sup_{t \geq N, v \in [\alpha - \epsilon, \alpha + \epsilon]} \mathbb{E}[\frac{t}{v \cdot t + m_{t+1}} \frac{m_{t+1}}{C}] < w_1/w < 1$, increasing N if necessary. (This is possible because for all v close enough to α and t large enough, the expectation is close to 1 as $\mathbb{E}[m_{t+1}/C] = \alpha$.) Let $D := \sup_{t \geq N, v \in [\alpha - \epsilon, \alpha + \epsilon]} \mathbb{E}[\frac{t}{v \cdot t + m_{t+1}} \frac{m_{t+1}}{C}]$. Let \mathcal{C}'_1 be the event that $|v(t) - \alpha| < \epsilon$ for all $t \geq T(N)$.

Suppose events \mathcal{C}_1 and \mathcal{C}'_1 hold and $\tau_n > T(n)$. Then we have

$$\begin{aligned} \sum_{t=\tau_n}^j h_1(\mathbf{y}(t))/(v(t) \cdot t) &= \sum_{t=\tau_n}^j \left\{ \mathbb{E}\left[\frac{v(t) \cdot t}{v(t) \cdot t + m_{t+1}} \frac{m_{t+1}}{C} \mid v(t)\right] \cdot (\phi_{\sigma, z(t)}(x(t)) - x(t)) \right\} / (v(t) \cdot t) \\ &\geq - \sum_{k=n}^{\infty} \frac{1}{T(k)^{1/2-\chi}} \sum_{T(k) \leq t < T(k+1)} \frac{\mathbb{E}[\frac{v(t) \cdot t}{v(t) \cdot t + m_{t+1}} \cdot \frac{m_{t+1}}{C} \mid v(t)]}{v(t) \cdot t} \text{ by the definition of } \mathcal{C}_1 \\ &\geq -D \cdot \sum_{k=n}^{\infty} \frac{\log(\lceil T(k+1) \rceil) - \log(\lceil T(k) \rceil)}{T(k)^{1/2-\chi}} \text{ by the definition of } \mathcal{C}'_1 \\ &\geq -D \cdot \sum_{k=n}^{\infty} \left(\frac{1-r_0}{\gamma w_1} + 1 \right) \cdot e^{-k(1/2-\chi)(1-r_0)/(\gamma w_1)} \\ &= -D \cdot \left(\frac{1-r_0}{\gamma w_1} + 1 \right) \cdot \frac{e^{-n(1/2-\chi)(1-r_0)/(\gamma w_1)}}{1 - e^{-(1/2-\chi)(1-r_0)/(\gamma w_1)}}. \end{aligned} \tag{15}$$

We define $\mu = D \cdot \left(\frac{1-r_0}{\gamma w_1} + 1 \right) \cdot \frac{1}{1 - e^{-(1/2-\chi)(1-r_0)/(\gamma w_1)}}$, so that the right-hand side is $-\mu T(n)^{-(1/2-\chi)}$.

Let \mathcal{C}_2 be the event that for all $n \geq N$ and for all $t \geq T(n)$,

$$\phi_{\sigma, z(t)}(x) - x \leq w\gamma r_0^n \tag{16}$$

for all $x \in [x^* - \gamma r_0^n, x^*]$. Because $\phi_{\sigma,z}(x) - x$ is polynomial (in z and x) and

$$\frac{d(\phi_{\sigma,q}(x) - x)}{dx}(x^*) = 0,$$

we can choose B' such that for all $n \geq N$ we have

$$\phi_{\sigma,z}(x) - x \leq w\gamma r_0^n$$

for $x \in [x^* - \gamma r_0^n, x^*]$ whenever $|z - q| < B'r_0^n$ (since we can bound the entries of the Hessian of $\phi_{\sigma,z}(x) - x$ above by a constant on the rectangle $[x^* - \gamma r_0^N, x^*] \times [q - r_0^N, q + r_0^N]$). Because $r_0^n > T(n)^{1/2-\chi}$, this holds for $|z(t) - q| < B'/T(n)^{1/2-\chi}$ for some $B' > 0$.

Define the event $\mathcal{C} = \mathcal{C}_1 \cap \mathcal{C}'_1 \cap \mathcal{C}_2$ to be the intersection of these three events. The event \mathcal{C} holds when $|z(t) - q| < \min(B, B')/T(n)^{1/2-\chi}$ for all $n \geq N$ and all $t \geq T(n)$. By the Chernoff bound and the inequalities $t \geq T(n)$ and $q > 1 - q$, the probability of $|z(t) - q| > \min(B, B')/T(n)^{1/2-\chi}$ is at most $2e^{-\min(B, B')^2 t^{2\chi}/(2q^2)}$. So the probability that the event \mathcal{C} does not hold for some $n \geq N$ and all $t \geq T(n)$ is at most

$$2 \sum_{n=N}^{\infty} \sum_{t=T(n)}^{\infty} 2e^{-\min(B, B')^2 t^{2\chi}/(2q^2)}.$$

For N sufficiently large, this sum is approximately

$$\sum_{n=N}^{\infty} \frac{1}{\chi} \left(\frac{\min(B, B')^2}{2q^2} \right)^{-\frac{1}{2\chi}} \Gamma \left(\frac{1}{2\chi}, T(n)^{2\chi} \min(B, B')^2/(2q^2) \right)$$

where $\Gamma(s, x)$ is the incomplete Gamma function. Since $\Gamma(s, x)/(x^{s-1}e^{-x}) \rightarrow 1$ as $x \rightarrow \infty$, this sum converges to zero as $N \rightarrow \infty$. Increasing N if necessary, we can conclude that the event $\mathcal{C}_1 \cap \mathcal{C}_2$ has positive probability. Clearly the same is also true for $\mathcal{C}_1 \cap \mathcal{C}'_1 \cap \mathcal{C}_2$, as $v(t)$ and $z(t)$ are independent. For the remainder of case (i), we condition on this event \mathcal{C} .

Now let \mathcal{B} be the event $\{\inf_{j > \tau_n} x(j) \geq x^* - \gamma r_0^n\}$. We will bound the probability of this event conditional on $\tau_n > T(n)$. Let $Z_{k,n} = \sum_{t=k}^{n-1} M(t+1)$ be the sum of the martingale parts of the stochastic process. Because the differences $M(t)$ are martingales with $|M(t)| \leq 1/(t \cdot (\alpha - \epsilon) + 1)$

on the event \mathcal{C}'_1 , we have

$$\mathbb{E}[Z_{k,\infty}^2] = \sum_{t=k}^{\infty} \mathbb{E}[M(t)^2] \leq \sum_{t=k}^{\infty} \left(\frac{1}{t \cdot (\alpha - \epsilon) + 1} \right)^2 \leq \frac{1}{k(\alpha - \epsilon)^2}. \quad (17)$$

We have:

$$\begin{aligned} \mathbb{P}[\mathcal{B}^c \mid \tau_n > T(n)] &= \mathbb{P} \left[\inf_{j > \tau_n} x(j) < x^* - \gamma r_0^n \mid \tau_n > T(n) \right] \\ &\leq \mathbb{P} \left[\inf_{j > \tau_n} Z_{\tau_n, j} < -(\gamma - 1)r_0^n + \mu T(n)^{-(1/2-\chi)} \mid \tau_n > T(n) \right] \text{ by equation (15)} \\ &\leq \mathbb{E} \left[Z_{\tau_n, \infty}^2 \mid \tau_n > T(n) \right] / ((\gamma - 1)r_0^n - \mu T(n)^{-(1/2-\chi)})^2 \text{ by Chebyshev's inequality} \\ &\leq (\alpha - \epsilon)^{-2} e^{-n(1-r_0)/(w_1 \gamma)} ((\gamma - 1)r_0^n - \mu T(n)^{-(1/2-\chi)})^{-2} \\ &\text{by inequality (17) and the definition of } T(n). \end{aligned}$$

Recall that $T(n)^{1/2-\chi} r_0^{-n} \rightarrow \infty$, so for n sufficiently large

$$(\gamma - 1)r_0^n - \mu T(n)^{-(1/2-\chi)} \geq \frac{\gamma - 1}{2} r_0^n.$$

We conclude that

$$\mathbb{P}[\mathcal{B}^c \mid \tau_n > T(n)] \leq (\alpha - \epsilon)^{-2} \left(\frac{\gamma - 1}{2} \right)^{-2} g(r_0)^{-2n}.$$

This bounds the conditional probability of the event \mathcal{B} not holding.

When the event \mathcal{B} does hold and $\tau_n > T(n)$,

$$\begin{aligned}
\sum_{\substack{T(n) < t < T(n+1) \\ x(t) < x^*}} h_1(\mathbf{y}(t))/(v(t) \cdot t) &= \sum_{\substack{T(n) < t < T(n+1) \\ x(t) < x^*}} \left\{ \mathbb{E} \left[\frac{v(t) \cdot t}{v(t) \cdot t + m_{t+1}} \frac{m_{t+1}}{C} \mid v(t) \right] \cdot (\phi_{\sigma, z(t)}(x(t)) - x(t)) \right\} / (v(t) \cdot t) \\
&\leq \sum_{\substack{T(n) < t < T(n+1) \\ x(t) < x^*}} Dw\gamma r_0^n / (t) \text{ by definition of } \mathcal{C}_1 \text{ and equation (16)} \\
&\leq (\log \lceil T(n+1) \rceil - \log \lceil T(n) \rceil) (Dw\gamma r_0^n) \\
&\text{by the partial sums of the harmonic series} \\
&\leq (Dw\gamma r_0^n) ((1 - r_0)/(\gamma w_1) + 1/T(n)) \\
&= (Dw/w_1)(r_0^n - r_0^{n+1}) + Dw\gamma r_0^n / T(n).
\end{aligned}$$

Now suppose \mathcal{B} holds and $\tau_n > T(n)$ but $\tau_{n+1} \leq T(n+1)$. Then

$$\begin{aligned}
Z_{\tau_n, \tau_{n+1}} &= x(\tau_{n+1}) - x(\tau_n) - \sum_{t=\tau_n}^{\tau_{n+1}-1} h_1(\mathbf{y}(t))/(v(t) \cdot t) \\
&\geq x(\tau_{n+1}) - x(\tau_n) - \sum_{\substack{T(n) < t < T(n+1) \\ x(t) < x^*}} h_1(\mathbf{y}(t))/(v(t) \cdot t) \\
&\geq r_0^n - r_0^{n+1} - \xi_n - D(w/w_1)(r_0^n - r_0^{n+1}) - Dw\gamma r_0^n / T(n) \text{ by the inequality above and definition of } \tau_n \\
&= r_0^n (1 - r_0)(1 - D(w/w_1)) - \xi_n - Dw\gamma r_0^n / T(n),
\end{aligned}$$

where ξ_n is an error term since $x(\tau_n)$ may be larger than $x^* - r_0^n$ and $\tilde{\xi}_n = \xi_n + w\gamma r_0^n / T(n)$. (Recall $0 < D(w/w_1) < 1$.) Since the error term ξ_n is at most $1/T(n)$ and therefore is lower order than r_0^n , we have

$$\frac{r_0^n (1 - r_0)(1 - Dw/w_1) - \tilde{\xi}_n}{r_0^n (1 - r_0)(1 - Dw/w_1)} \rightarrow 1 \tag{18}$$

as $n \rightarrow \infty$.

Combining our bounds, we have:

$$\begin{aligned}
\mathbb{P}[\tau_{n+1} \leq T(n+1) \mid \tau_n > T(n)] &\leq \mathbb{P}[\mathcal{B}^c \mid \tau_n > T(n)] + \\
&\mathbb{P}\left[\mathcal{B}, \sup_j Z_{\tau_n, j} \geq r_0^n(1-r_0)(1-Dw/w_1) - \tilde{\xi}_n \mid \tau_n > T(n)\right] \\
&\leq (\alpha - \epsilon)^{-2} \left(\frac{\gamma-1}{2}\right)^{-2} g(r_0)^{-2n} + \frac{\mathbb{E}[Z_{\tau_n, \infty}^2 \mid \tau_n > T(n)]}{(r_0^n(1-r_0)(1-Dw/w_1) - \tilde{\xi}_n)^2} \\
&\text{by Chebyshev's inequality} \\
&\leq (\alpha - \epsilon)^{-2} \left(\frac{\gamma-1}{2}\right)^{-2} g(r_0)^{-2n} + \frac{(\alpha - \epsilon)^{-2} T(n)^{-1}}{(r_0^n(1-r_0)(1-Dw/w_1) - \tilde{\xi}_n)^2} \\
&\text{by inequality (4)} \\
&\leq (\alpha - \epsilon)^{-2} \left(\frac{\gamma-1}{2}\right)^{-2} g(r_0)^{-2n} + \\
&(\alpha - \epsilon)^{-2} ((1-r_0)(1-Dw/w_1))^{-2} g(r_0)^{-2n} \cdot \frac{r_0^n(1-r_0)(1-Dw/w_1) - \tilde{\xi}_n}{r_0^n(1-r_0)(1-Dw/w_1)}.
\end{aligned}$$

We claim that the sum of these probabilities converges. The sum of the first terms converges because $g(r_0) > 1$. For the second term, recall that the fraction $\frac{r_0^n(1-r_0)(1-Dw/w_1) - \tilde{\xi}_n}{r_0^n(1-r_0)(1-Dw/w_1)}$ converges to 1. So the sum of the second terms also converges because $g(r_0) > 1$.

We have

$$\mathbb{P}[\tau_n > T(n) \text{ for all } n \geq N] = \mathbb{P}[\tau_N > T(N)] \prod_{n=N}^{\infty} (1 - \mathbb{P}[\tau_{n+1} \leq T(n+1) \mid \tau_n > T(n)]).$$

On the right-hand side, each factor in the product is positive and

$$\sum_{n=N}^{\infty} \mathbb{P}[\tau_{n+1} \leq T(n+1) \mid \tau_n > T(n)]$$

is finite. By a standard result on infinite products, this implies the product is positive. So the probability that $\tau_n > T(n)$ for all $n \geq N$ is positive, which implies that the probability $\pi(x^* \mid \sigma)$ of converging to x^* is positive.

Case (ii): There exists $\epsilon > 0$ such that $\phi_\sigma(x) > x$ for all $x \in (x^* - \epsilon, x^*)$ and $\phi_\sigma(x) < x$ for all $x \in (x^*, x^* + \epsilon)$.

We begin with a lemma, which says that suitably changing a stochastic process away from a

neighborhood of a fixed point does not affect whether we converge to that fixed point with positive probability.

Lemma 7. *Consider a process $\mathbf{y}(t) = (x(t), v(t), z(t))$ valued in $[0, 3]^3$ satisfying the conditions in the proof of Proposition 8. Suppose*

$$\tilde{\mathbf{y}}(t+1) = \tilde{\mathbf{y}}(t) + \frac{1}{\tilde{v}(t) \cdot t} (\tilde{\boldsymbol{\xi}}(t+1) - \tilde{\mathbf{y}}(t)),$$

where the conditionally i.i.d. random variables $\tilde{\boldsymbol{\xi}}(t+1)$ have the same conditional distributions as $\boldsymbol{\xi}(t+1)$ in a neighborhood U of (x^*, α, q) , have the same support as $\boldsymbol{\xi}(t+1)$ for all $(x, v, z) \in (0, 1)^3$, have expectations $\mathbb{E}[\tilde{\boldsymbol{\xi}}(t+1)]$ that are Lipschitz continuous in (x, v, z) , and with probability 1 $\sum_t \frac{1}{\tilde{v}(t) \cdot t} = \infty$ while $\sum_t \frac{1}{(\tilde{v}(t) \cdot t)^2} < \infty$. Then $x(t)$ converges to x^* with positive probability if and only if $\tilde{x}(t) = \tilde{\mathbf{y}}_1(t)$ does.

Proof. The argument is exactly analogous to that of the proof of Lemma 2. □

Now choose $\tilde{\boldsymbol{\xi}}(t)$ satisfying the conditions of Lemma 7, agreeing with $\boldsymbol{\xi}(t)$ in the second and third coordinates, and such that the unique fixed point of the corresponding $\tilde{\phi}_\sigma(x)$ is x^* . To do so, choose an open neighborhood U of (x^*, α, q) such that x^* is the unique fixed point of $\phi_\sigma(x)$ with $(x, v, q) \in \bar{U}$. Let $\tilde{\boldsymbol{\xi}}(t) = \boldsymbol{\xi}(t)$ on the closure \bar{U} of U . For each (v, z) , let $\tilde{\boldsymbol{\xi}}(t)$ be constant in x outside of the neighborhood U .

Then, since we have not modified the process $\tilde{\mathbf{y}}(t)$ in the second or third coordinates, we continue to have $\sum_t \frac{1}{\tilde{v}(t) \cdot t} = \infty$ while $\sum_t \frac{1}{(\tilde{v}(t) \cdot t)^2} < \infty$ with probability 1. Also, $\tilde{\boldsymbol{\xi}}(t)$ and $\boldsymbol{\xi}(t)$ have the same support because conditional on every interior (x, v, z) , $\boldsymbol{\xi}(t)$ has the same support. Therefore, making $\tilde{\boldsymbol{\xi}}(t)$ be constant in x does not affect its support. Lipschitz continuity follows from Lipschitz continuity of the expectation of $\boldsymbol{\xi}(t)$ in (x, v, z) , which we checked in the proof of Proposition 8.

Since x^* is the unique fixed point of $\phi_\sigma(x)$, by the same argument as in Proposition 8, we have $\tilde{x}(t) \rightarrow x^*$ almost surely. Note that this step uses Lipschitz continuity of $\mathbb{E}[\tilde{\boldsymbol{\xi}}(t+1)]$. So by Lemma 7, $x(t) \rightarrow x^*$ with positive probability. □

Finally, we need the following lemma to relate the inflow accuracy function induced by some strategy σ^* and viral accuracy process induced by strategies sufficiently close to σ^* .

Lemma 8. For each $\epsilon', \epsilon'' > 0$ $p \in (0, 1)$, strategy σ^* with $\phi_{\sigma^*}(x) - x \geq 2\epsilon'$ for every $x \leq p + 2\epsilon'$, there is some N and $\delta > 0$ so that for every σ with $\|\sigma - \sigma^*\|_2 < \delta$, we have $\mathbb{P}_\sigma[x(t) \geq p + \epsilon'/2] > 1 - \epsilon''$ for every $t \geq N$.

Proof. Let $\phi_{\sigma,z}(x) := \frac{\sum_{s \in \{-1,1\}} \sum_{k_R=0}^{K_R} \sum_{k_V=0}^{K_V} \mathbb{E}[\sigma(s, k_R, k_V)] \cdot \mathbb{P}[s|\omega=1] \cdot \mathbb{P}[\text{Binom}(K_R, z) = k_R] \cdot \mathbb{P}[\text{Binom}(K_V, x) = k_V]}{C}$.

Because $\phi_{\sigma,z}(x)$ is polynomial in z , σ , and x , there exists $\delta > 0$ such that $\phi_{\sigma,z}(x) - x \geq \epsilon'$ for every $x \leq p + \epsilon'$ when $\|\sigma^* - \sigma\|_2 < \delta$ and $|z - q| < \delta$.

For the remainder of the proof, fix σ in this neighborhood. We will observe at the end of the proof that the bounds we will prove are uniform in the choice of σ .

Let $p' > p + \epsilon'$ be the largest number in $(0, 1)$ such that

$$\phi_\sigma(x) - x \geq \epsilon'/2 \quad (19)$$

for all $x \leq p'$. Let $N_1 < N_2$ be positive integers with $N_2 \geq bN_1$ for some integer $b > 1$. We will first show that for N_1 and N_2 large enough, the probability that $x(t) < p'$ for all $t \in [N_1, N_2]$ is small. We will then show that if $x(t_1) > p'$ for some $N_1 \leq t_1 < N_2$, then the probability that $x(N_2) < p + \epsilon'/2$ is small.

By the Chernoff bound applied to $z(t)$ and compactness of the set of strategies σ under consideration, we can choose a constant $B > 0$ independent of σ such that

$$\max_{x \in [0,1]} |\phi_{\sigma,z(t)}(x) - \phi_\sigma(x)| < \epsilon'/4 \quad (20)$$

with probability at least $1 - 2e^{-Bt}$ for t sufficiently large.

Make ϵ' smaller if necessary so that $0 < \alpha - \epsilon' < \alpha + \epsilon' < 1$. For constants $T, D > 0$, define the event

$$\mathcal{C} = \inf_{t \geq T} \mathbb{E}\left[\frac{t}{v \cdot t + m_{t+1}} \frac{m_{t+1}}{C}\right] > D \text{ and } \sup_{t \geq T} |v(t) - \alpha| < \epsilon'.$$

Since $v(t) \rightarrow \alpha$ almost surely and since this process is unaffected by σ and independent of $x(t), z(t)$, we may find $T, D > 0$ independent of σ such that $\mathbb{P}[\mathcal{C}] \geq 1 - \epsilon''/2$.

Recall that we can decompose $\mathbf{y}(t)$ as

$$\mathbf{y}(t+1) = \mathbf{y}(t) + \frac{1}{v(t) \cdot t} \cdot h(\mathbf{y}(t)) + M(t+1),$$

where $h(\mathbf{y}(t))$ is deterministic and $M(t+1)$ is a martingale. As in the proof of Theorem 4, $h_1(\mathbf{y}(t)) = \mathbb{E}[\frac{v(t) \cdot t}{v(t) \cdot t + m_{t+1}} \frac{m_{t+1}}{C} \mid v(t)] \cdot (\phi_{\sigma, z(t)}(x(t)) - x(t))$. Also, we have $|M(t)| \leq 1/(t \cdot (\alpha - \epsilon) + 1)$ for all $t \geq T$ on the event \mathcal{C} . So by Theorem C.7 from Appendix C of Borkar (2023), for any $\beta > 0$ and any t_1 and t_2 ,

$$\mathbb{P} \left(\sup_{t_1 < t < t_2} \left| \sum_{i=t_1}^t M(i) \right| > \beta \right) \leq 4e^{-\frac{\beta^2}{\sum_{i=t_1}^{t_2} (1/(\alpha - \epsilon))^2 / i^2}}. \quad (21)$$

Consider the event E that $x(t) < p'$ for all $N_1 \leq t \leq N_2$. Suppose inequality (20) holds for all $N_1 \leq t < N_2$ and condition on the event \mathcal{C} . Then we have, when $N_1, N_2 \geq T$,

$$\begin{aligned} x(N_2) - x(N_1) &= \sum_{t=N_1}^{N_2-1} \mathbb{E}[\frac{v(t) \cdot t}{v(t) \cdot t + m_{t+1}} \frac{m_{t+1}}{C} \mid v(t)] \cdot \frac{\phi_{\sigma, z(t)}(x(t)) - x(t)}{v(t) \cdot t} + \sum_{t=N_1}^{N_2-1} M(t+1) \\ &= \sum_{t=N_1}^{N_2-1} \mathbb{E}[\frac{t}{v(t) \cdot t + m_{t+1}} \frac{m_{t+1}}{C} \mid v(t)] \left\{ \frac{\phi_{\sigma, z(t)}(x(t)) - \phi_{\sigma}(x(t))}{t} + \frac{\phi_{\sigma}(x(t)) - x(t)}{t} \right\} + \sum_{t=N_1}^{N_2-1} M(t+1) \\ &\geq \sum_{t=N_1}^{N_2-1} D \cdot \epsilon' / 4 \cdot \frac{1}{t} + \sum_{t=N_1}^{N_2-1} M(t+1) \text{ by inequalities (19) and (20)} \\ &\geq D(\epsilon' / 4)(\log(N_2) - \log(N_1)) + \sum_{t=N_1}^{N_2-1} M(t+1). \end{aligned}$$

When event E holds, the right-hand side must be at most p' . Taking b and therefore N_2/N_1 sufficiently large, we can assume that

$$D(\epsilon' / 4)(\log(N_2) - \log(N_1)) > 2p'.$$

By equation (21), the absolute value of the sum of martingales is greater than p' with probability at most

$$4e^{-\frac{(p')^2}{\sum_{i=N_1}^{N_2} (1/(\alpha - \epsilon))^2 / i^2}} \leq 4e^{-\frac{(p')^2 N_1 N_2}{2(1/(\alpha - \epsilon))^2 \cdot (N_2 - N_1)}} < 4e^{-\frac{(p')^2 N_1}{2(1/(\alpha - \epsilon))^2}}.$$

Along with the Chernoff bound, this gives an upper bound on the probability of event E given \mathcal{C} .

If event E does not hold, there exists some $N_1 \leq t \leq N_2$ such that $x(t) \geq p'$. Choose t_1 so that $t_1 - 1$ is the largest such t .

Suppose $x(N_2) \leq p + \epsilon' / 2$. For N_1 sufficiently large, this implies $t_1 \leq N_2$. So we must have

$$x(N_2) - x(t_1) \leq ((p + \epsilon' / 2) - p') < -\epsilon'.$$

On the other hand, when inequality (20) holds for all $N_1 \leq t < N_2$ and conditional on \mathcal{C} , for $N_1, N_2 \geq T$ we have

$$\begin{aligned}
x(N_2) - x(t_1) &= \sum_{t=t_1}^{N_2-1} \mathbb{E}\left[\frac{v(t) \cdot t}{v(t) \cdot t + m_{t+1}} \frac{m_{t+1}}{C} \mid v(t)\right] \cdot \frac{\phi_{\sigma, z(t)}(x(t)) - x(t)}{v(t) \cdot t} + \sum_{t=t_1}^{N_2-1} M(t+1) \\
&= \sum_{t=t_1}^{N_2-1} \mathbb{E}\left[\frac{t}{v(t) \cdot t + m_{t+1}} \frac{m_{t+1}}{C} \mid v(t)\right] \cdot \left(\frac{\phi_{\sigma, z(t)}(x(t)) - \phi_{\sigma}(x(t))}{t} + \frac{\phi_{\sigma}(x(t)) - x(t)}{t}\right) + \sum_{t=t_1}^{N_2-1} M(t+1) \\
&\geq \sum_{t=t_1}^{N_2-1} D \cdot \epsilon'/4 \cdot \frac{1}{t} + \sum_{t=t_1}^{N_2-1} M(t+1) \text{ by inequalities (19) and (20)} \\
&\geq D \cdot (\epsilon'/4)(\log(N_2) - \log(t_1)) + \sum_{t=t_1}^{N_2-1} M(t+1).
\end{aligned}$$

Applying equation (21) with $\beta = \epsilon'$, the absolute value of the sum of martingales is greater than ϵ' with probability at most

$$4e^{-\frac{(\epsilon')^2}{\sum_{i=t_1}^{N_2} (1/(\alpha-\epsilon))^2 / i^2}} \leq 4e^{-\frac{(p')^2 t_1}{2(1/(\alpha-\epsilon))^2}}.$$

When this does not hold and the Chernoff bounds apply, $x(N_2) - x(t_1)$ is greater than $-\epsilon'$ and therefore $x(N_2) > p + \epsilon'/2$ if N_1 is sufficiently large. This gives an upper bound on the probability that $x(N_2) \leq p + \epsilon'/2$.

We conclude that

$$\mathbb{P}_{\sigma}[x(N_2) < p + \epsilon'/2 \mid \mathcal{C}] \leq 4e^{-\frac{(p')^2 N_1}{2(1/(\alpha-\epsilon))^2}} + \sum_{t=N_1+1}^{N_2-1} 4e^{-\frac{(p')^2 t}{2(1/(\alpha-\epsilon))^2}} + 2 \sum_{t=N_1}^{N_2-1} e^{-Bt}$$

for N_1 sufficiently large. Because the second and third terms are geometric series, we can choose N_1 sufficiently large so that the right hand side is less than $\epsilon''/2$ for all $N_2 \geq bN_1$. We can make this choice uniformly in σ (subject to the constraint $\|\sigma - \sigma^*\|_2 < \delta$). Since $\mathbb{P}_{\sigma}[\mathcal{C}] \geq 1 - \epsilon''/2$, for N_1 sufficiently large, we have

$$\mathbb{P}_{\sigma}[x(t) \geq p + \epsilon'/2] > 1 - \epsilon''$$

for $t \geq N = bN_1$. □

Finally, we present the proof of Theorem 3.

Proof. Step 1: There exists \bar{K}_V so that whenever $K_V \geq \bar{K}_V$, there is no limit equilibrium where all steady states are strictly larger than q .

Let $1/2 + \eta$ be the probability that $\mathbb{P}[\text{Binom}(K_R, 1-q) \geq C]$, so we have $\eta > 0$ by the hypothesis.

Find \bar{K}'_V large enough so that $\lfloor 0.5\eta \cdot \bar{K}'_V \rfloor > K_R + 1$. By Chebyshev's inequality, find \bar{K}''_V large enough so that $\mathbb{P}[\text{Binom}(\bar{K}''_V, 0.5 + 0.5\eta) > \bar{K}''_V \cdot (0.5 + 0.25\eta)] > 1 - 0.5\eta$. Let $\bar{K}_V = \max(\bar{K}'_V, \bar{K}''_V)$.

For $K_V \geq \bar{K}_V$, in any limit equilibrium σ^* where σ^* follows the majority of the viral news feed whenever there is a consensus of size larger than $0.5 + 0.25\eta$, there is a misleading steady state. To see this, we bound $\phi_{\sigma^*}(0.5 - 0.5\eta)$. Since $K_V \geq \bar{K}''_V$, there is more than $1 - 0.5\eta$ chance that there are more than $0.5 + 0.25\eta$ fraction of wrong stories in the viral news feed. When this happens, σ^* must share C wrong stories from the regular news feed in the event that there are at least C wrong stories there, which happens with probability at least $1/2 + \eta$. So, $\phi_{\sigma^*}(0.5 - 0.5\eta) < 0.5\eta + (1/2 - \eta) = 0.5 - 0.5\eta$. As $\phi_{\sigma^*}(0) > 0$, $\phi_{\sigma^*}(0.5 - 0.5\eta) < 0.5 - 0.5\eta$, and $\phi_{\sigma^*}(\cdot)$ is a polynomial function, we conclude there must be a fixed point in $(0, 0.5 - 0.5\eta)$ that is stable from both sides. This fixed point is a steady state by Theorem 4.

It now suffices to show that in a limit equilibrium σ^* where $K_V \geq \bar{K}_V$ and all steady states are strictly larger than q , the equilibrium strategy follows the majority of the viral news feed whenever there is a consensus of size larger than $0.5 + 0.25\eta$. Because the lowest steady state induced by σ^* is strictly larger than q , we may find $\epsilon' > 0$ so that $\phi_{\sigma^*}(x) - x \geq 2\epsilon'$ for every $x \leq q + 2\epsilon'$. Let \mathcal{O} be the finite class of all observations where there is a majority of size at least $0.5 + 0.25\eta$ in the viral news feed. For each $o \in \mathcal{O}$, if the agent has a prior belief about the distribution of viral accuracy that assigns probability 1 to the segment $[q + \epsilon'/2, 1]$, then the posterior belief assigns probability strictly more than 0.5 to the state of nature matching the viral news feed majority. This is because $K_V \geq \bar{K}'_V$ (so $\lfloor 0.5\eta \cdot K_V \rfloor > K_R + 1$) and because viral news feed stories are more precise than regular news feed stories. By continuity of Bayesian updating, the same must also hold when the prior assigns probability less than $h_o > 0$ to the complement of the segment $[q + \epsilon'/2, 1]$. Let $h = \min_{o \in \mathcal{O}} h_o > 0$. Applying Lemma 8 with ϵ' , $\epsilon'' = h/2$, and $p = q$, we find N and $\delta > 0$ such that for every σ with $\|\sigma - \sigma^*\|_2 < \delta$, we have $\mathbb{P}_\sigma[x(t) \geq q + \epsilon'/2] > 1 - h/2$ for every $t \geq N$. In the sequence of converging equilibria $\sigma^{(j)} \rightarrow \sigma^*$, eventually $\|\sigma^{(j)} - \sigma^*\|_2 < \delta$. Also, the number of agents grows so that eventually, $N/n_j < h/2$. So in equilibria $\sigma^{(j)}$ with large enough j , agents (who have uniformly random beliefs about their location in the sequence) have prior beliefs about

viral accuracy that assign probability more than $1 - h$ to $[q + \epsilon'/2, 1]$. So $\sigma^{(j)}$ follows the viral news feed majority for all observations in \mathcal{O} . The same must then hold for the limit σ^* .

Step 2: There is no limit equilibrium where $1/2$ is a steady state.

We will first define an important class of strategies. For a strategy σ and for $0 < k_R < K_R$, say $\sigma(s_i, k_R, k_V)$ is *maximally negative* if it is sharing as many negative stories from the regular news feed as possible, *maximally positive* if it is sharing as many positive stories from the regular news feed as possible, and *strictly mixing* if it is neither maximally negative nor maximally positive. Say a strategy σ is a *cutoff strategy* if for every fixed \bar{k}_V , there exists a cutoff $\kappa \in \{0, 1, 2, \dots, K_R\}$ so that $\sigma(s_i, k_R, \bar{k}_V)$ is maximally negative whenever $k_R + \mathbf{1}\{s_i = 1\} < \kappa$, maximally positive whenever $k_R + \mathbf{1}\{s_i = 1\} > \kappa$, and it is not the case that $\sigma(s_i, k_R, \bar{k}_V)$ is maximally negative for every k_R, s_i pair such that $k_R + \mathbf{1}\{s_i = 1\} = \kappa$. So the cutoff $\kappa(\bar{k}_V)$ is the minimum required number of positive signals of precision q to switch the strategy from being maximally negative to either being maximally positive or mixing between the two kinds of stories, after the agent sees \bar{k}_V positive stories in the viral news feed. When $\kappa = 0$, the strategy $\sigma(\cdot, \cdot, \bar{k}_V)$ is maximally positive for all realizations of s_i and k_R . When $\kappa = K_R$, the strategy $\sigma(\cdot, \cdot, \bar{k}_V)$ is maximally negative for all realizations of s_i and k_R .

We need the following lemma:

Lemma 9. *Every limit equilibrium is a cutoff strategy.*

Proof. Consider any converging sequence of equilibria $(\sigma^{(j)})$, where the equilibrium $\sigma^{(j)}$ is for a society with n_j agents and $n_j \rightarrow \infty$. In the society with n_j agents, suppose the agent observes viral news story realizations $s_1^V, \dots, s_{K_V}^V$ and $1 \leq k \leq K + 1$ total positive stories out of the regular news story feed and private signal. We show that for all j large enough, the posterior belief in $\{\omega = 1\}$ following this observation is strictly higher than that following the same observation but with only $k - 1$ positive stories in the regular news feed and the private signal. This would show that σ^j is a cutoff strategy for all j large enough, which means the limit must also be a cutoff strategy.

For any sufficiently small $\epsilon > 0$, the probability that $(x(\tau), z(\tau))$ at a uniformly random position τ in equilibrium $\sigma^{(j)}$ is in $[\epsilon, 1 - \epsilon] \times [q - \epsilon, q + \epsilon]$ converges to 1 as $j \rightarrow \infty$. This is because $\phi_\sigma(x) - x$ is uniformly bounded above 0 for all x close enough to 0 and $\phi_\sigma(x) - x$ is uniformly bounded below 0 for all x close enough to 1, across all strategies, so applying Lemma 8 shows

that $\mathbb{P}_{\sigma^j}[x(\tau) \in [\epsilon, 1 - \epsilon]] \rightarrow 1$ as $j \rightarrow \infty$ if $\epsilon > 0$ is small enough. The fact that we must have $\mathbb{P}_{\sigma^j}[z(\tau) \in [q - \epsilon, q + \epsilon]] \rightarrow 1$ as $j \rightarrow \infty$ follows from the law of large numbers.

Let A be the event that $(x(\tau), z(\tau)) \in [\epsilon, 1 - \epsilon] \times [q - \epsilon, q + \epsilon]$. Let $f^{(j)}(x|A)$ be the discrete distribution of viral accuracy in society j at a random position, conditional on the event A . Also using the fact that $z(\tau) \in [q - \epsilon, q + \epsilon]$ conditional on A , we get

$$\begin{aligned} & \mathbb{P}_{\sigma^j}[s_1^V, \dots, s_{K_V}^V, k, A \mid \omega = 1] \\ & \geq \mathbb{P}_{\sigma^j}[A \mid \omega = 1] \cdot (q - \epsilon)^k \cdot (1 - q - \epsilon)^{K-k} \int \prod_{k_V=1}^{K_V} x^{1\{s_{k_V}^V=1\}} \cdot (1-x)^{1\{s_{k_V}^V=-1\}} f^j(x \mid A) dx. \end{aligned} \quad (22)$$

$$\begin{aligned} & \mathbb{P}_{\sigma^j}[s_1^V, \dots, s_{K_V}^V, k, A \mid \omega = -1] \\ & \leq \mathbb{P}_{\sigma^j}[A \mid \omega = -1] \cdot (1 - q + \epsilon)^k \cdot (q + \epsilon)^{K-k} \int \prod_{k_V=1}^{K_V} (1-x)^{1\{s_{k_V}^V=1\}} \cdot x^{1\{s_{k_V}^V=-1\}} f^j(x \mid A) dx. \end{aligned} \quad (23)$$

$$\begin{aligned} & \mathbb{P}_{\sigma^j}[s_1^V, \dots, s_{K_V}^V, k-1, A \mid \omega = 1] \\ & \leq \mathbb{P}_{\sigma^j}[A \mid \omega = 1] \cdot (q + \epsilon)^{k-1} \cdot (1 - q + \epsilon)^{K-k+1} \cdot \int \prod_{k_V=1}^{K_V} x^{1\{s_{k_V}^V=1\}} \cdot (1-x)^{1\{s_{k_V}^V=-1\}} f^j(x \mid A) dx. \end{aligned} \quad (24)$$

$$\begin{aligned} & \mathbb{P}_{\sigma^j}[s_1^V, \dots, s_{K_V}^V, k-1, A \mid \omega = -1] \\ & \geq \mathbb{P}_{\sigma^j}[A \mid \omega = -1] \cdot (1 - q - \epsilon)^{k-1} \cdot (q - \epsilon)^{K-k+1} \cdot \int \prod_{k_V=1}^{K_V} (1-x)^{1\{s_{k_V}^V=1\}} \cdot x^{1\{s_{k_V}^V=-1\}} f^j(x \mid A) dx. \end{aligned} \quad (25)$$

We have

$$\frac{\mathbb{P}_{\sigma^j}[s_1^V, \dots, s_{K_V}^V, k \mid \omega = 1]}{\mathbb{P}_{\sigma^j}[s_1^V, \dots, s_{K_V}^V, k \mid \omega = -1]} / \frac{\mathbb{P}_{\sigma^j}[s_1^V, \dots, s_{K_V}^V, k-1 \mid \omega = 1]}{\mathbb{P}_{\sigma^j}[s_1^V, \dots, s_{K_V}^V, k-1 \mid \omega = -1]} \quad (26)$$

$$= \frac{\mathbb{P}_{\sigma^j}[s_1^V, \dots, s_{K_V}^V, k, A \mid \omega = 1] + \mathbb{P}_{\sigma^j}[s_1^V, \dots, s_{K_V}^V, k, A^c \mid \omega = 1]}{\mathbb{P}_{\sigma^j}[s_1^V, \dots, s_{K_V}^V, k-1, A \mid \omega = 1] + \mathbb{P}_{\sigma^j}[s_1^V, \dots, s_{K_V}^V, k-1, A^c \mid \omega = 1]} \quad (27)$$

$$\times \frac{\mathbb{P}_{\sigma^j}[s_1^V, \dots, s_{K_V}^V, k-1, A \mid \omega = -1] + \mathbb{P}_{\sigma^j}[s_1^V, \dots, s_{K_V}^V, k-1, A^c \mid \omega = -1]}{\mathbb{P}_{\sigma^j}[s_1^V, \dots, s_{K_V}^V, k, A \mid \omega = -1] + \mathbb{P}_{\sigma^j}[s_1^V, \dots, s_{K_V}^V, k, A^c \mid \omega = -1]} \quad (28)$$

Using (22) and (24), the term (27) is at least

$$\frac{(q - \epsilon)^k (1 - q - \epsilon)^{K-k} \{\mathbb{P}_{\sigma^j}[A \mid \omega = 1] \int \prod_{k_V=1}^{K_V} x^{1\{s_{k_V}^V=1\}} \cdot (1-x)^{1\{s_{k_V}^V=-1\}} f^j(x \mid A) dx\} + \mathbb{P}_{\sigma^j}[s_1^V, \dots, s_{K_V}^V, k, A^c \mid \omega = 1]}{(q + \epsilon)^{k-1} (1 - q + \epsilon)^{K-k+1} \{\mathbb{P}_{\sigma^j}[A \mid \omega = 1] \int \prod_{k_V=1}^{K_V} x^{1\{s_{k_V}^V=1\}} \cdot (1-x)^{1\{s_{k_V}^V=-1\}} f^j(x \mid A) dx\} + \mathbb{P}_{\sigma^j}[s_1^V, \dots, s_{K_V}^V, k-1, A^c \mid \omega = 1]}.$$

For $\epsilon > 0$ small enough, we have $(q - \epsilon)^k \cdot (1 - q - \epsilon)^{K-k} > (q + \epsilon)^{k-1} \cdot (1 - q + \epsilon)^{K-k+1}$ and also $\mathbb{P}_{\sigma^j}[A \mid \omega = 1] \cdot \int \prod_{k_V=1}^{K_V} x^{1\{s_{k_V}^V=1\}} \cdot (1-x)^{1\{s_{k_V}^V=-1\}} f^j(x \mid A) dx$ is bounded away from 0 across all j . Also, for any $\epsilon > 0$, $\mathbb{P}_{\sigma^j}[s_1^V, \dots, s_{K_V}^V, k, A^c \mid \omega = 1] \rightarrow 0$ and $\mathbb{P}_{\sigma^j}[s_1^V, \dots, s_{K_V}^V, k-1, A^c \mid \omega = 1] \rightarrow 0$ as $j \rightarrow \infty$. This shows that for all $\epsilon > 0$ small enough, the term (27) is strictly larger than 1 for all large enough j . We can similarly use (23) and (25) to show that for $\epsilon > 0$ small enough, the term (28) is strictly larger than 1 for all large enough j . Thus we conclude for all large enough j , $\frac{\mathbb{P}_{\sigma^j}[s_1^V, \dots, s_{K_V}^V, k \mid \omega = 1]}{\mathbb{P}_{\sigma^j}[s_1^V, \dots, s_{K_V}^V, k \mid \omega = -1]} / \frac{\mathbb{P}_{\sigma^j}[s_1^V, \dots, s_{K_V}^V, k-1 \mid \omega = 1]}{\mathbb{P}_{\sigma^j}[s_1^V, \dots, s_{K_V}^V, k-1 \mid \omega = -1]} > 1$ as needed. \square

In any limit equilibrium, since σ^* is the limit of cutoff strategies, σ^* must be a cutoff strategy. In a steady state with viral accuracy $1/2$, the distribution of stories in the viral news feed is the same conditional on either state of the world. For any fixed realization of the viral news feed, the expected number of positive regular news stories shared by σ^* is weakly increasing in $k_R + \mathbf{1}\{s_i = 1\}$. In fact, it must be strictly increasing somewhere, since σ^* shares zero positive regular stories when $k_R = 0, s_i = -1$ and shares C positive regular stories when $k_R = K_R, s_i = 1$. Since $q > 1/2$, the distribution of $k_R + \mathbf{1}\{s_i = 1\}$ in the state $\omega = 1$ first-order stochastically dominates its distribution in the state $\omega = -1$. Thus, the number of shared positive regular news stories is strictly positively correlated with ω , so $\phi_{\sigma^*}(1/2) > 1/2$.

Step 3: There is no limit equilibrium where all steady states are strictly larger than $1/2$ but not all steady states are strictly larger than q .

By way of contradiction, suppose such a limit equilibrium σ^* exists whose lowest steady state

is p with $1/2 < p \leq q$.

An implication of Lemma 9 is that σ^* is a cutoff strategy. We can show that it also has more structure. When $\bar{k}_V > K_V/2$, we must have $\kappa \leq \left\lceil \frac{K_R+1}{2} \right\rceil$. When $\bar{k}_V < K_V/2$, we must have $\kappa \geq \left\lceil \frac{K_R+1}{2} \right\rceil$. When $\bar{k}_V = K_V/2$, the cutoff is $\kappa = \left\lceil \frac{K_R+1}{2} \right\rceil$. To see why the first restriction holds (the other ones are symmetric), note that if an agent's belief about the distribution over viral accuracy puts probability 1 on the segment $[1/2 + \epsilon'/2, 1]$, the agent must think $\omega = 1$ is strictly more likely when there is a strictly majority of positive stories in the viral news feed and a majority of positive stories among the private signal and the regular news feed stories. Thus, such an agent would use a cutoff no larger than $\left\lceil \frac{K_R+1}{2} \right\rceil$. By an argument using Lemma 8 similar to the one in **Step 1**, we can show that in the converging sequence of equilibria $\sigma^{(j)} \rightarrow \sigma^*$, the same also holds for $\sigma^{(j)}$ for all j large enough.

Now consider $\phi_\sigma(p)$ for various cutoff strategies σ . This accuracy is maximized by a cutoff strategy that computes the Bayesian posterior probabilities of the two states of nature after every observation (treating stories in the viral news feed as signals with precision p) and maximally shares the stories in the direction of the more likely state. Let the optimal cutoff after seeing k_V positive viral news feed stories be $\kappa_{k_V}^{\text{opt}}$. We argue that accuracy after seeing k_V positive viral news feed stories is single-peaked in cutoff choice, with the peak at $\kappa_{k_V}^{\text{opt}}$. For $\kappa' \geq \kappa_{k_V}^{\text{opt}}$, compare the behavior given by the cutoff κ' and the cutoff $\kappa' + 1$. These two cutoffs imply the same behavior for $k_R + \mathbf{1}\{s_i = 1\} \geq \kappa' + 2$ and $k_R + \mathbf{1}\{s_i = 1\} \leq \kappa' - 1$, and thus have the same accuracy in those cases. When $k_R + \mathbf{1}\{s_i = 1\} = \kappa' + 1$, the state $\{\omega = 1\}$ is strictly more likely since $\kappa' + 1 > \kappa_{k_V}^{\text{opt}}$. Using the cutoff κ' leads to maximally positive sharing in this case, which cannot be improved. When $k_R + \mathbf{1}\{s_i = 1\} = \kappa'$, the state $\{\omega = 1\}$ is weakly more likely since $\kappa' \geq \kappa_{k_V}^{\text{opt}}$. Using the cutoff $\kappa' + 1$ leads to maximally negative sharing in this case, which cannot be made worse. This comparison thus shows the cutoff κ' must generate weakly higher accuracy than cutoff $\kappa' + 1$. Similarly, we can show that if $\kappa \leq \kappa_{k_V}^{\text{opt}}$, then cutoff κ generates weakly higher accuracy than cutoff $\kappa - 1$.

This shows the number of correct stories shared by σ^* at viral accuracy p and conditional on some number k_V of positive viral news feed stories is bounded by the conditional accuracy of the two extremal cutoff behaviors. For $k_V > K_V/2$, we know that the cutoff used by σ^* is between 0 and $\left\lceil \frac{K_R+1}{2} \right\rceil$, which respectively correspond to always maximally sharing in the positive direction and always following the majority among the private signal and the regular news feed stories. The

conditional accuracy of the cutoff strategy with cutoff $\left\lceil \frac{K_R+1}{2} \right\rceil$ is the accuracy of the majority among $K_R + 1$ signals of precision q , which is some number strictly higher than q .

We now bound the conditional accuracy of the cutoff strategy with cutoff 0. The conditional probability of $\{\omega = 1\}$ given $k_V > K_V/2$ positive viral news stories is some number $p' \geq p > 1/2$, with $p' > p$ if $k_V \geq (K_V/2) + 1$. Consider the j th regular news story shared for $1 \leq j \leq C$. This will be positive, except when $k_R \leq j - 1$. The event $\{k_R \leq j - 1\}$ happens with some probability ζ' when $\omega = 1$ and some probability ζ'' when $\omega = -1$. The probability that the j th regular story shared matches the state is therefore $p' \cdot (1 - \zeta') + (1 - p') \cdot \zeta''$.

For $p' \leq q$, this expression is no smaller than p' . This is because $j \leq C \leq K_R/2$, so we have $\zeta'/\zeta'' \leq (1 - q)/q$ which implies $\zeta'' \geq \frac{q}{1-q}\zeta'$. Making this substitution we find that accuracy is no smaller than $p' + \zeta' \cdot (\frac{q}{1-q} \cdot (1 - p') - p') \geq p' + \zeta' \cdot (\frac{q}{1-q}(1 - q) - q) = p'$, where we used $p' \leq q$ in the last inequality.

As p' increases beyond q , the expression is bounded between its value when $p' = q$ and its value when $p' = 1$. The former is at least p' , and the latter is $1 - \zeta'$. But ζ' is no larger than the probability that the regular news feed majority is wrong, which is $1 - q''$ for some $q'' > q$. Hence, $1 - \zeta' \geq q''$. So for any value of p' , $p' \cdot (1 - \zeta') + (1 - p') \cdot \zeta'' \geq \min(p', q'')$.

Thus, the conditional accuracy of the σ^* strategy given any $k_V > K_V/2$ is bounded below by $\min(p, q'')$ (since $p' \geq p$ whenever $k_V > K_V/2$), and it is bounded below by some $\min(p', q'')$ for $p' > p$ for any $k_V \geq K_V/2 + 1$. An analogous argument applies to the case of $k_V < K_V/2$. So overall the average accuracy of this strategy is strictly higher than $\min(p, q'') \geq p$. This is a contradiction as it shows $\phi_{\sigma^*}(p) > p$. \square

B Details of the Equilibrium Simulations for $\lambda > \lambda^*$

In these simulations, we fix virality weight $\lambda = 1$, story precision $q = 0.55$, and sharing capacity $C = 3$. We consider three different values of the news-feed size, $K \in \{6, 8, 10\}$.

For each of these three parameter specifications, we first consider all candidate symmetric pure-strategy limit equilibria. Under any symmetric pure strategy, the likelihood ratio of $\omega = 1$ to $\omega = -1$ after observing $k = K/2$ positive news-feed stories is 1, whereas the likelihood ratio after observing $k = K/2 + 1, K/2 + 2, \dots, K$ positive stories is the reciprocal of the likelihood ratio after

observing $K - k$ such stories. For each $k \in \{K/2 + 1, \dots, K\}$, the likelihood ratio falls into one of the following three cases: (1) between $\frac{1-q}{q}$ and $\frac{q}{1-q}$, so it is optimal for the agent to follow their private signal; (2) below $\frac{1-q}{q}$, so it is optimal for the agent to share as many negative stories as possible; (3) above $\frac{q}{1-q}$, so it is optimal for the agent to share as many positive stories as possible. Each of the $3^{K/2}$ assignments of these three cases to various values of $k \in \{K/2 + 1, \dots, K\}$ implies a best-responding strategy, and we conduct 5,000 repetitions of a numerical simulation with 5,000 agents to check whether the likelihood ratios of the various observations generated by this strategy indeed belong to the same cases. In this way, we find no pure-strategy limit equilibrium for $K = 6$ and $K = 8$, and we find one pure-strategy limit equilibrium as described in the main text.

We next look for symmetric mixed-strategy limit equilibria with the following structure: the likelihood ratio after observing $K/2 + 1$ positive stories is exactly equal to $\frac{q}{1-q}$, whereas the likelihood ratio after observing $k > K/2 + 1$ positive stories is strictly above $\frac{q}{1-q}$. This requires the agent to follow their private signal with some probability $p \in [0, 1]$ when they see $K/2 + 1$ positive stories. For each of $K \in \{6, 8, 10\}$, we consider all mixed strategies with the values $p = 0, 0.05, 0.10, \dots, 0.95, 1.0$. For each such mixed strategy, we conduct 10,000 repetitions of a numerical simulation with 10,000 agents. These simulations suggest that no such mixed equilibria exist for $K = 10$, for the likelihood ratio associated with a news feed with 6 positive stories is always significantly below $\frac{q}{1-q}$ for any p in the grid we considered. For $K = 6$ and $K = 8$, we instead find that the likelihood ratio associated with a news feed with $K/2 + 1$ positive stories is below $\frac{q}{1-q}$ for low values of p but above $\frac{q}{1-q}$ for high values of p , suggesting the existence of a mixed limit-equilibrium. Zooming in on the segment $p \in [0, 0.4]$ for $K = 6$ and the segment $p \in [0.6, 1]$ for $K = 8$, as suggested by the initial set of simulations, we conduct further simulations with the values of p on a grid with width 0.02. For each strategy, we conduct 30,000 repetitions of a numerical simulation with 20,000 agents. These simulations allow us to estimate the equilibrium in a society with t agents for each $t \in \{200, 201, \dots, 20000\}$ by linearly interpolating the value of p that would set the likelihood ratio of an observation with $K/2 + 1$ positive stories to be exactly $\frac{q}{1-q}$. These estimated mixing probabilities in finite-society equilibria are shown in Figure 4 (solid curves).

Then, to estimate limit equilibria, we use non-linear least squares to fit a rational function of the form $t \mapsto \frac{at+b}{ct+1}$ to approximate the equilibrium mixing probability p_t in a society with t agents. The best-fitting rational functions are shown as dashed curves in Figure 4, which fit very closely

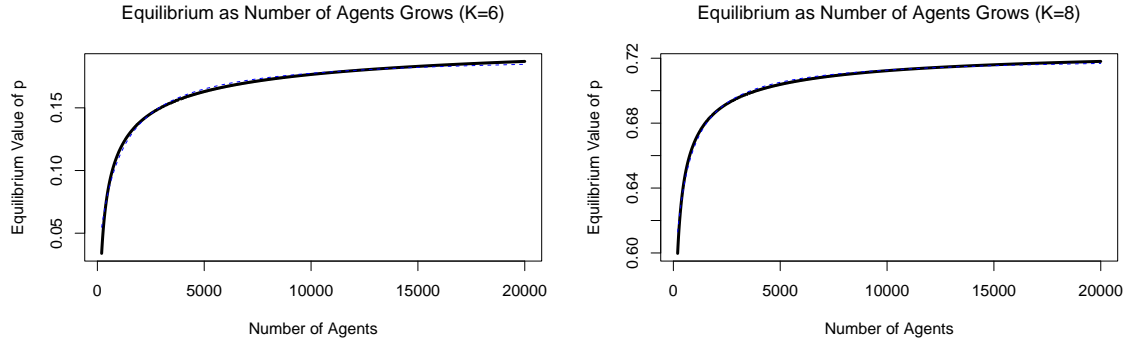


Figure 4: Estimated equilibrium mixing probabilities for different population sizes (solid curves) and estimated rational function (dashed curves).

to the estimated mixing probabilities. We divide the estimated coefficients a and c in the rational function to estimate $\lim_{t \rightarrow \infty} p_t$. This procedure estimates $\lim_{t \rightarrow \infty} p_t \approx 0.192605$ for $K = 6$ and $\lim_{t \rightarrow \infty} p_t \approx 0.7211451$ for $K = 8$. The inflow accuracy functions of these two limit strategies in their respective environments are plotted in Figure 5. For $K = 8$, the misleading steady state resembles a touchpoint.

Finally, we conduct a final set of numerical simulations to estimate the long-run distribution of viral accuracy under the limit equilibrium strategies. For each of $K = 6, 8, 10$, we conduct 10,000 repetitions of a numerical simulation with 40,000 agents, using the estimated limit equilibrium strategies. The distributions of viral accuracy by period 40,000 are shown in Figure 6, where we see faster convergence to a steady state for the $K = 10$ pure-strategy limit equilibrium than the $K = 6$ and $K = 8$ mixed-strategy limit equilibria.

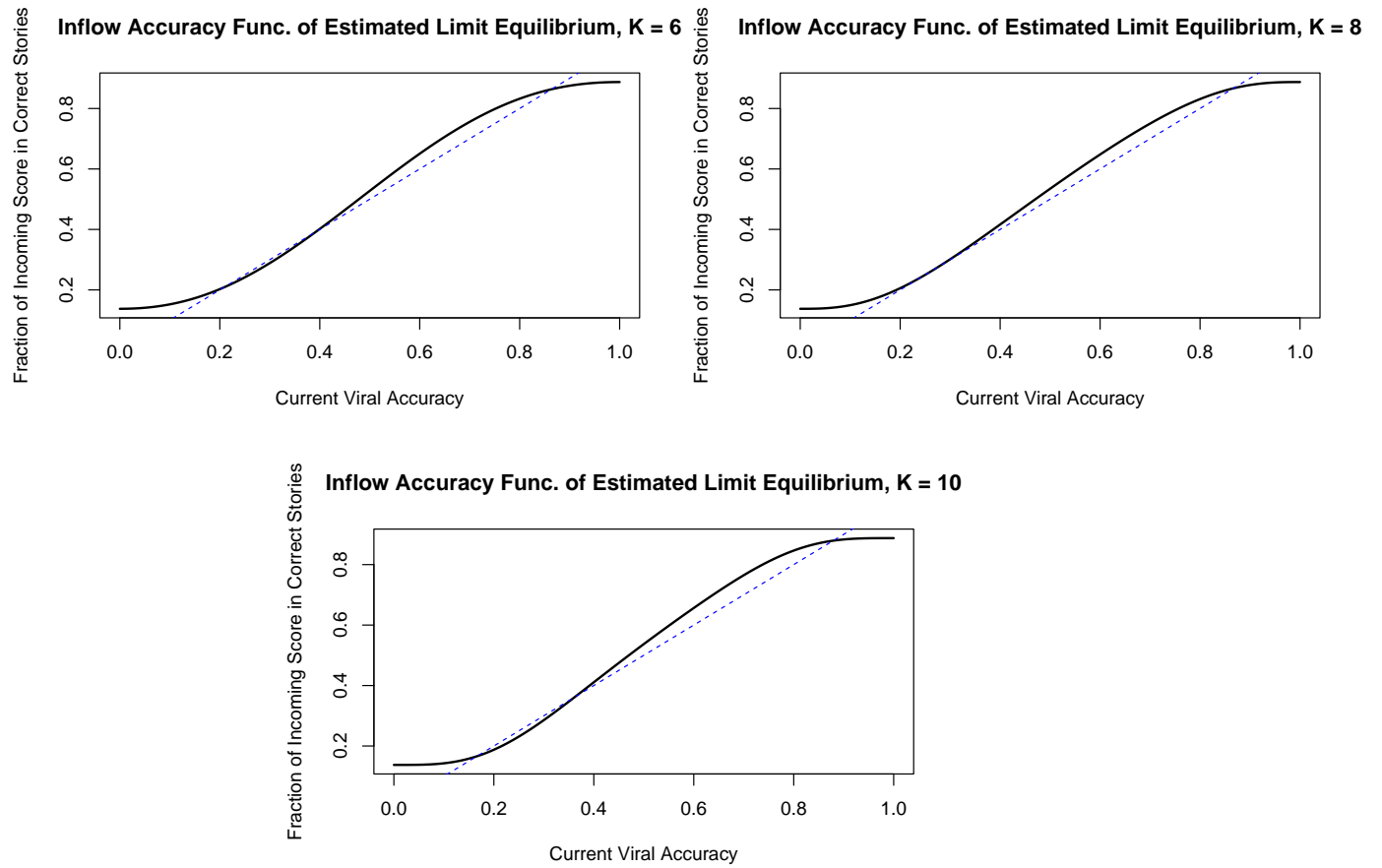


Figure 5: Inflow accuracy functions of the estimated limit equilibria.

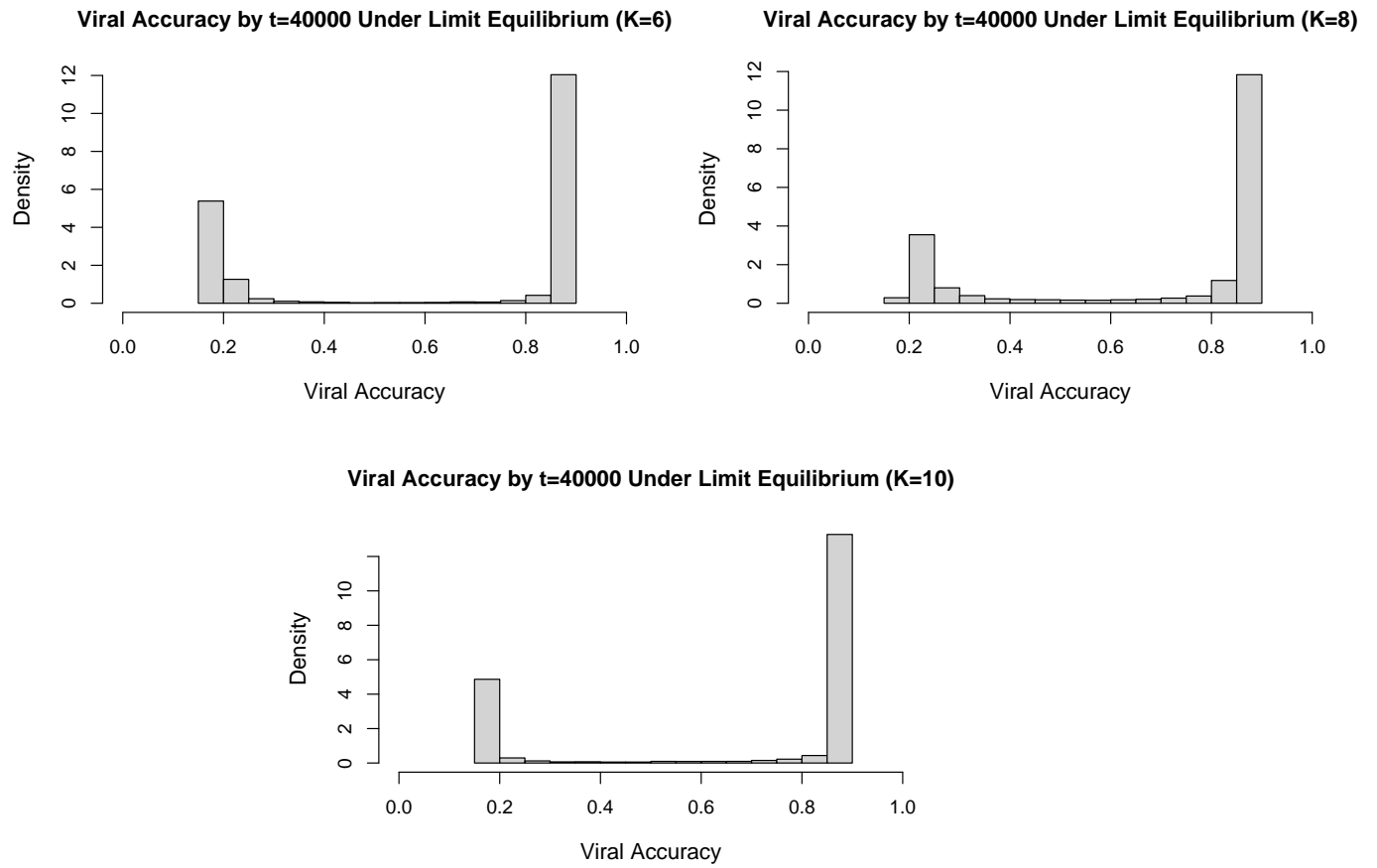


Figure 6: Distributions of viral accuracy by period 40,000 under estimated limit equilibria. Each histogram shows results from 10,000 simulations.